



Elastodynamics of two-dimensional odd mechanical media: Theory, homogenization and wave characterization

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ABSTRACT

Non-Hermitian systems have attracted growing attention in mechanics due to their unique ability to manipulate energy in unconventional ways. In this study, we establish a comprehensive theoretical framework to explore exotic wave phenomena in odd elasticity. Specifically, we reveal polarization-dependent energy transfer in the unbroken-phase, second-order amplification at exceptional points (EPs), and exponential energy gain or loss in the broken-phase. To substantiate these findings, we perform an energy-based analysis within a homogeneous medium, offering clear physical interpretations that corroborate our theoretical predictions. Furthermore, we develop a homogenization theory to examine the elastodynamic behavior of various two-dimensional (2D) odd lattices constructed from tensile-torsional asymmetric springs. This theory is validated through comparisons of dispersion relations and coupled mode shapes with those obtained from microstructural simulations. The energy-transfer characteristics and directional wave amplification observed in the lattices closely align with the predictions of the homogenized model. Finally, we demonstrate unidirectional surface wave propagation and mode conversion as striking manifestations of non-Hermitian behavior in mechanical systems. This work provides a broadly applicable framework for investigating non-Hermitian effects in elastic wave systems and introduces novel strategies for the precise manipulation and control of mechanical energy.

1. Introduction

Elastic wave control has emerged as a foundational strategy in advanced material engineering (Cummer et al., 2016), underpinning key technologies such as vibration isolation (Casadei et al., 2012; Shan et al., 2014), acoustic noise suppression (Yang et al., 2008; Fang et al., 2006; Christensen and De Abajo, 2012), and structural health monitoring (Farrar and Worden, 2007). These capabilities are crucial for enhancing the performance, longevity, and resilience of mechanical systems across a broad spectrum of engineering applications. Historically, elastic metamaterials composed of carefully engineered microstructures have enabled unprecedented control over wave propagation (Kadic et al., 2013; Cheng et al., 2024), leading to remarkable phenomena such as negative refraction (Zhu et al., 2014a), low-frequency bandgaps (Xiao et al., 2012; Wang et al., 2014), and the development of elastic metasurfaces

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(Li et al., 2018; Su et al., 2018). However, these conventional metamaterials operate under the constraints of passivity, reciprocity, and energy conservation (Maznev et al., 2013; Trainiti and Ruzzene, 2016), which inherently limit their adaptability and functionality, particularly in dynamic or non-equilibrium settings. To transcend these limitations, recent research has shifted focus to active metamaterials (Yi et al., 2019; Chen et al., 2017, 2016, 2014), which incorporate external energy sources or embedded actuation to introduce non-conservative dynamics. These active systems unlock new mechanical behaviors that are inaccessible within classical elasticity (Chen et al., 2020), including non-reciprocal wave transmission (Nassar et al., 2017; Zhao et al., 2020; Cheng and Hu, 2021), gain-loss modulation (Fleury et al., 2015; Zhu et al., 2014b), and real-time tunability (Trainiti et al., 2019; Liu et al., 2024). At the core of many of these breakthroughs lies the theoretical framework of non-Hermitian physics (Ashida et al., 2020; Christensen et al., 2024), which allows for complex-valued eigenfrequencies and the emergence of exceptional points (EPs) in the system's dynamic spectrum (Geng et al., 2021; Heiss, 2012). This marks a paradigm shift from classical Hermitian models, enabling fundamentally new approaches to wave manipulation, mode control, and energy flux in mechanical systems (Geng et al., 2024; Scheibner et al., 2020a).

The emergence of non-Hermitian mechanics, characterized by the interplay of gain and loss and by the breakdown of reciprocity, challenges the foundations of classical continuum theories, which are inherently conservative and reciprocal. This growing field opens new frontiers in wave control, energy manipulation, and directional mechanical behavior, yet it also exposes a critical theoretical gap: conventional elasticity cannot account for the complex, often asymmetric responses observed in non-Hermitian systems. To address this discrepancy, researchers have developed generalized homogenization techniques and extended continuum models that link microscale structural architectures to macroscopic mechanical behavior. These models serve as essential tools for the inverse design of non-Hermitian mechanical metamaterials, enabling targeted control over wave propagation and stiffness asymmetry. Notable approaches include odd density media characterized by non-symmetric mass density tensors to model anomalous inertia effects (Wang et al., 2024a; Wu et al., 2023); chiral beam models incorporating asymmetric moment-shear coupling for directional bending behavior (Wang et al., 2024b); and odd plate theories featuring non-symmetric bending stiffness matrices (Chen et al., 2021). Collectively, these methods extend classical elasticity by introducing asymmetric moduli and complex-valued constitutive parameters, offering a robust theoretical foundation for exploring and engineering non-Hermitian wave phenomena (Wu et al., 2024). Among these, odd elasticity stands out as a particularly powerful framework (Scheibner et al., 2020b). By introducing an antisymmetric component into the stress-strain relationship, it allows for directional stiffness and internal energy exchange mechanisms that inherently violate reciprocity and energy conservation. As a continuum theory, odd elasticity enables the prediction and control of unconventional mechanical responses in structured media. Despite significant promise, a comprehensive theoretical framework for odd elastodynamics, the dynamic extension of odd elasticity, has yet to be fully explored. Equally important is the development of a systematic homogenization theory that can effectively connect microstructural design to the emergence of non-symmetric stiffness at the continuum level. Addressing this fundamental gap is essential not only for advancing our understanding of dynamic non-Hermitian behavior but also for unlocking novel capabilities in programmable wave manipulation within artificial mechanical materials.

In this study, we develop a comprehensive theoretical framework for elastodynamics for 2D odd elastic media, a representative and foundational class within non-Hermitian continuum mechanics. These behaviors challenge traditional wave propagation theories based on material symmetry by incorporating antisymmetric components into their constitutive relations, thereby enabling non-reciprocal, path-dependent mechanical responses. Unlike conventional elastic materials, where stress derives symmetrically from strain and energy is conserved under cyclic deformation, odd elastic media exhibit directional stiffness and intrinsic energy exchange with their surroundings. As a result, the mechanical response becomes sensitive to the order and spatial distribution of applied deformations, producing novel wave behaviors such as asymmetric dispersion, wave amplification, and unidirectional energy transport. The goal of this paper is twofold: first, to establish a rigorous theoretical foundation for the continuum description of odd elasticity; and second, to demonstrate its utility in predicting and designing complex wave phenomena in structured media. By doing so, we aim to bridge the gap between microscale lattice design and macroscale dynamic response, paving the way for inverse design methodologies that are currently lacking in non-Hermitian mechanics. Moreover, although our analysis is formulated in 2D, the main results extend naturally to three-dimensional (3D) odd elastic media, where the deformation modes become more intricate due to the presence of nine strain components and their associated stresses. The structural design principles and analytical framework developed in this work can be readily adapted to guide future investigations of 3D odd elasticity. The structure of the paper is as follows. [Section 2](#) introduces a geometric interpretation of antisymmetric stress, rooted in a higher dimensional formulation of Stokes' theorem, which offers both physical intuition and mathematical clarity on the emergence of odd elastic effects. This section also examines how different non-Hermitian parameters, including antisymmetric moduli and complex coupling terms, govern wave propagation characteristics. [Section 3](#) constructs a homogenized continuum model derived from a discrete lattice of odd springs, capturing key features such as broken major symmetry and non-reciprocal force transmission. This model is rigorously validated through comparisons with discrete numerical simulations, confirming its predictive capability across the full spectrum of dynamic behavior. [Section 4](#) explores surface wave steering and bulk mode conversion enabled by odd elasticity, revealing mechanisms that enable wavefront shaping, energy localization, and non-reciprocal mode coupling, which are not attainable within classical elastic frameworks. Finally, [Section 5](#) concludes with a forward-looking discussion on the broader impact of non-Hermitian continuum theories. We highlight their potential to guide the inverse design of active mechanical metamaterials with functionalities such as wave-based sensing, directional energy focusing, and mechanical signal processing. These capabilities remain unattainable within traditional Hermitian frameworks, underscoring the necessity of odd elasticity as a core design principle for next-generation mechanical systems operating in nonequilibrium regimes.

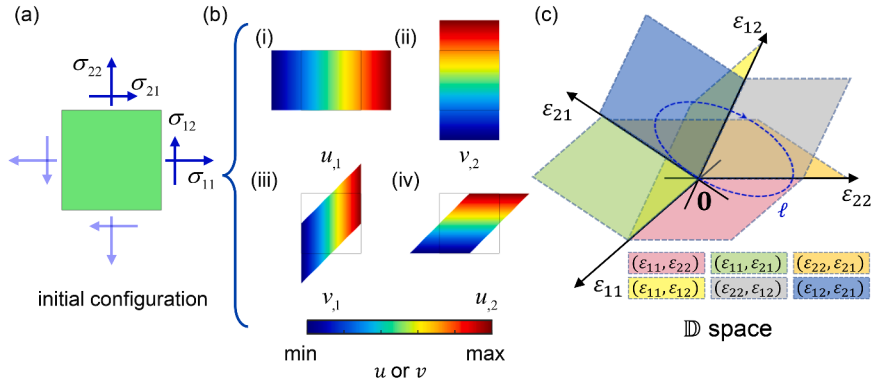


Fig. 1. Overview of fundamental concepts in 2D odd elasticity. (a) Schematic of an infinitesimal material element in the undeformed state. (b) Representation of the strain tensor (or displacement gradient) in 2D, showing (i, ii) volumetric strains and (iii, iv) shape (shear) strains. The color map indicates the value of displacement. (c) Illustration of the path-dependent work performed by the odd elastic medium along a closed trajectory in the 4D synthetic strain space $\mathbb{D}$.

2. Elastodynamic modelling of 2D odd elastic media

In this section, we develop a comprehensive elastodynamic model for 2D odd elastic media, following the framework established in Scheibner et al. (2020b). This model forms the theoretical basis for analyzing the non-conservative mechanical characteristics and unique dynamic responses of such materials.

2.1. Formulation of 2D odd elasticity

Odd elasticity describes a class of elastic media in which the strain energy cannot be derived from a conservative potential. In such systems, the mixed partial derivatives of the strain energy with respect to the strain components do not commute, thereby violating the symmetry conditions traditionally assumed in classical Cauchy elasticity (Lurie, 2005). To capture this distinctive behavior, we first present the fundamental physical equations governing odd elastic media. These include the geometric relations, constitutive laws, and governing equations, which together form the foundation for the subsequent analysis.

In the planar (2D) setting, an infinitesimal material element is illustrated in Fig. 1(a). The associated stress tensor contains four independent components. The notation σ_{ij} denotes the stress component acting in the j -direction on a surface whose outward normal is aligned with the i -direction. This definition highlights the intrinsic asymmetry in stress transmission that characterizes odd elastic systems.

In classical Cauchy elasticity, the rotational components of deformation are treated as rigid-body motions and, therefore, do not contribute to the strain energy. However, this assumption breaks down in active elastic systems, where embedded energy sources or control mechanisms introduce non-conservative effects. As a result, the conventional geometric relations must be modified. The revised form is given by:

$$\epsilon_{ij} = u_{j,i}. \quad (1)$$

In this equation, the comma denotes partial differentiation with respect to the spatial coordinates, and indices $i = 1, 2$ refer to the horizontal (u) and vertical (v) displacement components, respectively. Within this framework, the strain tensor is defined as the left gradient of the displacement field, which naturally includes both symmetric (strain) and antisymmetric (rotation) components. These components are schematically illustrated in Fig. 1(b), which also distinguishes between volumetric (panels (i) and (ii)) and shear strain modes (panels (iii) and (iv)), where the displacement component is encoded by the background color according to the accompanying color bar.

The general planar constitutive relation for odd elasticity is expressed as $\sigma_{ij} = K_{ijkl}\epsilon_{kl}$, where K_{ijkl} is the fourth-order elasticity tensor. In Voigt notation, this relationship can be reformulated as:

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \\ \sigma_{21} \end{bmatrix} = \begin{bmatrix} K_{11} & K_{12} & K_{13} & K_{14} \\ K_{21} & K_{22} & K_{23} & K_{24} \\ K_{31} & K_{32} & K_{33} & K_{34} \\ K_{41} & K_{42} & K_{43} & K_{44} \end{bmatrix} \begin{bmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \epsilon_{12} \\ \epsilon_{21} \end{bmatrix}. \quad (2)$$

Although the presence of active sources leads to a breakdown of the major symmetry in the elasticity matrix, the underlying constitutive relation remains consistent with causality and satisfies the Kramers-Kronig relations, as derived in Appendix A. To elucidate the non-conservative characteristics inherent to active elasticity, we decompose the elasticity matrix $\mathbf{K}$ into its symmetric and anti-symmetric components:

$$\mathbf{K}^S = (\mathbf{K} + \mathbf{K}^T)/2, \quad \mathbf{K}^A = (\mathbf{K} - \mathbf{K}^T)/2, \quad (3)$$

where the superscripts “ S ” and “ A ” denote the symmetric and antisymmetric components, respectively, and “ T ” indicates the matrix transpose. In a similar manner, the stress vector is defined as: $\sigma = [\sigma_{11}, \sigma_{22}, \sigma_{12}, \sigma_{21}]^T$, which can likewise be decomposed into symmetric and antisymmetric components, corresponding to those of the elasticity matrix: $\sigma^S = \mathbf{K}^S \cdot \varepsilon$, and $\sigma^A = \mathbf{K}^A \cdot \varepsilon$. This decomposition highlights the emergence of reveals non-conservative stress components arising from the antisymmetric part of the elasticity matrix, which characterizes odd elasticity and distinguishes it from classical elastic theories.

To analyze how strain energy evolves during material deformation, we introduce a synthetic strain space. In this formulation, the components of the strain tensor ε_{ij} span a four-dimensional(4D) vector space, defined as: $\mathbb{D} = \text{span}\{\varepsilon_{11}, \varepsilon_{22}, \varepsilon_{12}, \varepsilon_{21}\}$. This representation enables a detailed examination of energy variation over the course of a deformation cycle. A visual depiction of this synthetic strain space is provided in Fig. 1(c), where each axis corresponds to an individual strain component. To illustrate the non-conservative behavior inherent in odd elasticity, we consider an arbitrary evolutionary strain path ℓ within the space $\mathbb{D}$. The area manifold enclosed by the closed curve ℓ is denoted by $\mathbb{A}_\ell$, which represents the oriented projected areas of the path onto the 2D planes. It consists of six independent components associated with the following plane pairs: $(\varepsilon_{11}, \varepsilon_{22})$, $(\varepsilon_{11}, \varepsilon_{12})$, $(\varepsilon_{11}, \varepsilon_{21})$, $(\varepsilon_{22}, \varepsilon_{12})$, $(\varepsilon_{22}, \varepsilon_{21})$ and $(\varepsilon_{12}, \varepsilon_{21})$. The explicit calculation of these components is provided in Appendix B.3. Within this framework, Stokes’ theorem (Gaffney, 1954) provides a powerful tool for evaluating the mechanical energy associated with a closed strain path. The total energy can be expressed as:

$$\begin{aligned} E_{\text{Homo}} &= \oint_\ell \sigma^T \cdot d\varepsilon = \oint_\ell ((\sigma^A)^T + (\sigma^S)^T) \cdot d\varepsilon \\ &= \oint_{\mathbb{D}_\ell} (\nabla_\varepsilon \times \sigma^S) \cdot d\mathbf{A}_\varepsilon + \oint_{\mathbb{D}_\ell} (\nabla_\varepsilon \times \sigma^A) \cdot d\mathbf{A}_\varepsilon. \end{aligned} \quad (4)$$

Here, $\mathbb{D}_\ell$ denotes the region enclosed by the oriented closed curve ℓ in the synthetic strain space, with $\ell = \partial\mathbb{D}_\ell$. In this formulation, the strain space $\mathbb{D}$ is treated as a differentiable 4D manifold, which allows the application of tools from differential geometry. Specifically, the exterior derivative of the stress vector can be expressed in terms of the partial derivatives of the elasticity matrix, as given by the component-wise expressions below:

$$\begin{aligned} (\nabla_\varepsilon \times \sigma^S)_{ij} &= \frac{\partial \sigma^S(j)}{\partial \varepsilon(i)} - \frac{\partial \sigma^S(i)}{\partial \varepsilon(j)} = \mathbf{K}^S(j, i) - \mathbf{K}^S(i, j) = 0 \\ (\nabla_\varepsilon \times \sigma^A)_{ij} &= \frac{\partial \sigma^A(j)}{\partial \varepsilon(i)} - \frac{\partial \sigma^A(i)}{\partial \varepsilon(j)} = \mathbf{K}^A(j, i) - \mathbf{K}^A(i, j) = -2\mathbf{K}^A(i, j) \end{aligned} \quad (5)$$

In this context, (i) and (i, j) refer to the indices of vector and matrix components, respectively. As Eq. (5) shows, the symmetric part of the elasticity matrix does not contribute to the net energy over a closed path. Instead, it is the antisymmetric component that governs the emergent non-conservative behavior. The corresponding energy is thus given by $E_{\text{Homo}} = -2 \sum_{i \neq j, i, j=1 \sim 4} \mathbf{K}^A(i, j) \mathbf{A}_\varepsilon(i, j)$, which highlights the geometric origin of odd elasticity, with the energy determined by an area integral over the strain-space manifold.

2.2. Wave propagation in non-Hermitian odd elastic media

Odd elasticity introduces path-dependence in the strain energy, representing a fundamental departure from classical elastic theory and enabling a broad spectrum of unconventional dynamic behaviors. To investigate the implications of such non-conservative mechanics for elastic wave propagation, we consider a homogeneous odd elastic medium as a representative case. This subsection focuses on the derivation and analysis of the governing equations and the dispersion characteristics of elastic waves in this medium. The fundamental equation describing wave motion in a homogeneous odd elastic material is given by:

$$\sigma_{ij,i} = -\omega^2 \rho u_j, \quad (6)$$

here, ρ denotes the material density. By substituting Eq. (2) into the above governing equation and assuming a plane wave form: $u_j = U_j e^{-i\mathbf{k} \cdot \mathbf{r} + i\omega t}$, Eq. (6) can be rewritten compactly in matrix form as:

$$\mathbf{D} \cdot \mathbf{U} = c_p^2 \mathbf{U}, \quad (7)$$

where $c_p = \omega/k$ is the phase velocity, and the wave vector is defined as $\mathbf{k} = k(\cos \theta, \sin \theta)^T$. Here, θ represents the wave propagation angle measured relative to the Cartesian coordinate system $\mathbf{r} = (x, y)^T = r(\cos \theta, \sin \theta)^T$. The dynamic matrix $\mathbf{D}$, which governs wave propagation, is explicitly given by:

$$\mathbf{D} = \begin{bmatrix} D_{11}^S & D_{12}^S + D_{12}^A \\ D_{12}^S - D_{12}^A & D_{22}^S \end{bmatrix}. \quad (8)$$

Here, the components D_{ij}^X explicitly depend on the elements of the elasticity tensor and are defined as follows:

$$\begin{aligned} \rho D_{11}^S &= K_{11}^S \cos^2 \theta + 2K_{13}^S \cos \theta \sin \theta + K_{44}^S \sin^2 \theta \\ \rho D_{12}^S &= K_{13}^S \cos^2 \theta + (K_{12}^S + K_{34}^S) \cos \theta \sin \theta + K_{24}^S \sin^2 \theta \\ \rho D_{22}^S &= K_{33}^S \cos^2 \theta + 2K_{23}^S \cos \theta \sin \theta + K_{22}^S \sin^2 \theta \\ \rho D_{12}^A &= K_{13}^A \cos^2 \theta + (K_{12}^A - K_{34}^A) \cos \theta \sin \theta - K_{24}^A \sin^2 \theta \end{aligned} \quad (9)$$

For convenience, we introduce the discriminant defined as: $\Delta = (D_{11}^S - D_{22}^S)^2 + 4(D_{12}^S - D_{12}^A)^2$. Using this discriminant, the complex dispersion relation, which determines the squared phase velocities c_p^2 and the associated eigenvectors $\psi_{1,2}$, can be succinctly expressed

as follows:

$$c_p^2 = \frac{(D_{11}^S + D_{22}^S) \pm \sqrt{\Delta}}{2},$$

$$\psi_{1,2} = \begin{bmatrix} D_{11}^S - D_{22}^S \pm \sqrt{\Delta} \\ 2(D_{12}^S - D_{12}^A) \end{bmatrix}. \quad (10)$$

For subsequent analysis, the eigenvectors are normalized as $\Phi_i = \psi_i / \sqrt{\psi_i^\dagger \cdot \psi_i}$, where the superscript $\dagger$ denotes the conjugate transpose operation. As revealed by Eq. (10), the non-Hermitian nature of the system manifests itself through two fundamental features: (1) Complex eigenvalues, which correspond physically to directional wave amplification or attenuation; (2) Non-orthogonal eigenvectors, signifying intrinsic mode coupling and anisotropic energy flux. These distinctive wave characteristics and their dependence on material parameters and elastic anisotropy will be explored in greater detail in the subsequent subsection.

We begin by analyzing the dynamic response of the active system within the odd elasticity framework. An arbitrary planar unit excitation vector $\mathbf{u}_p$ (with $|\mathbf{u}_p| = 1$) can be expressed in terms of the normalized eigenmode basis as follows:

$$\mathbf{u}_p = \sum_{i=1,2} c_i \Phi_i, \quad (11)$$

where c_i ($i = 1, 2$) represent the corresponding modal coefficients. These coefficients can be explicitly determined by projecting $\mathbf{u}_p$ onto the eigenbasis, yielding:

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 & \Phi_1^\dagger \cdot \Phi_2 \\ \Phi_2^\dagger \cdot \Phi_1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} \Phi_1^\dagger \cdot \mathbf{u}_p \\ \Phi_2^\dagger \cdot \mathbf{u}_p \end{bmatrix}. \quad (12)$$

For convenience, we introduce the parameter $\lambda = \Phi_1^\dagger \cdot \Phi_2$, noting that its complex conjugate is given by $\lambda^* = \Phi_2^\dagger \cdot \Phi_1$. Additionally, we define $p_i = \Phi_i^\dagger \cdot \mathbf{u}_p$ as the projection of the excitation vector onto each eigenmode. Here, the superscript $*$ denotes complex conjugation. Expanding the inverse matrix explicitly, we obtain:

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \frac{1}{1 - \lambda\lambda^*} \begin{bmatrix} 1 & -\lambda \\ -\lambda^* & 1 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \end{bmatrix}. \quad (13)$$

Consequently, the displacement field induced by this excitation in the odd elastic medium can be expressed as:

$$\mathbf{u}_f = \sum_{i=1,2} c_i \Phi_i e^{-i\mathbf{k} \cdot \mathbf{r} + i\omega t}, \quad (14)$$

for a given propagation direction, this expression can be reformulated by explicitly substituting the modal coefficients c_i as follows:

$$\mathbf{u}_f = \frac{e^{i\omega t}}{1 - \lambda\lambda^*} ((p_1 - \lambda p_2)\Phi_1 e^{-ik_1 r} + (-\lambda^* p_1 + p_2)\Phi_2 e^{-ik_2 r}). \quad (15)$$

Subsequently, the kinetic energy density along the propagation direction, defined as $W_T = -\frac{\rho\omega^2}{2} \mathbf{u}_f^\dagger \cdot \mathbf{u}_f$, can be expanded as:

$$W_T = -\frac{\rho\omega^2}{2(1 - \lambda\lambda^*)^2} \begin{pmatrix} (p_1 - \lambda p_2)(p_1^* - \lambda^* p_2^*) e^{2\text{Im}(k_1)r} + (-\lambda^* p_1 + p_2)(-\lambda p_1^* + p_2^*) e^{2\text{Im}(k_2)r} \\ + (p_1 - \lambda p_2)(-\lambda p_1^* + p_2^*) \lambda^* e^{i(k_2^* - k_1)r} + (-\lambda^* p_1 + p_2)(p_1^* - \lambda^* p_2^*) \lambda e^{i(k_1^* - k_2)r} \end{pmatrix}. \quad (16)$$

In a Hermitian (Cauchy) elastic medium, the eigenmodes are mutually orthogonal, leading to $\lambda = 0$. Under this condition, the modal coefficients simplify to $c_i = p_i$, and consequently, the kinetic energy density remains constant during wave propagation, given explicitly by $W_T = -\rho\omega^2/2$. In contrast, a nonzero λ in a non-Hermitian system results in more complex dynamics characterized by energy exchange between eigenmodes. This intricate interplay and its physical implications will be systematically examined in the following sections.

2.2.1. Case 1: Real eigenvalues and non-orthogonal eigenvectors (Unbroken-phase)

When $\Delta > 0$, the eigenvalue problem given by Eq. (10) yields two distinct real eigenvalues associated with non-orthogonal eigenvectors. In this unbroken-phase regime, both the wave numbers $k_{1,2}$ and the eigenmode overlap parameter λ are real valued, leading to a coherent superposition of eigenmodes. Under these conditions, the kinetic energy density simplifies to:

$$W_T = -\frac{\rho\omega^2}{2(1 - \lambda^2)} \begin{pmatrix} 1 + \lambda^2 - 2\lambda \text{Re}(p_1 p_2^*) + 2\lambda (\text{Re}(p_1 p_2^*) - \lambda) \cos((k_2 - k_1)r) \\ - 2\lambda \text{Im}(p_1 p_2^*) \sin((k_2 - k_1)r) \end{pmatrix}. \quad (17)$$

Here, $\text{Re}(\cdot)$ and $\text{Im}(\cdot)$ denote the real and imaginary parts, respectively. This expression reveals a cosine-type modulation of the energy density along the propagation direction, resulting from interference between two non-orthogonal eigenmodes. Notably, the amplitude and phase of this modulation strongly depend on the choice of excitation vector $\mathbf{u}_p$. This underscores how the interplay between modal overlap and relative phase differences governs the redistribution of wave energy in the unbroken-phase.

2.2.2. Case 2: EP

The EP marks a critical transition at which both eigenvalues and eigenvectors coalesce as system parameters vary continuously. This corresponds to the degenerate case $\Delta = 0$, where the dynamic matrix becomes defective and no longer possesses a complete set of linearly independent eigenvectors. At the EP, the system exhibits the following limiting behaviors:

$$\begin{aligned} \text{Wave vector: } k_{1,2} &\rightarrow k_{EP} \\ \text{Projection coefficients: } p_{1,2} &\rightarrow p_{EP} \\ \text{Asymmetry of dynamic matrix: } D_{12}^A &\rightarrow D_{EP} \end{aligned} \quad (18)$$

To evaluate the kinetic energy density in the vicinity of the EP, we perform a Taylor expansion of the interference terms: $\cos((k_2 - k_1)r) \approx 1 - \frac{1}{2}(k_2 - k_1)^2 r^2$ and $\sin((k_2 - k_1)r) \approx (k_2 - k_1)r$. Substituting these approximations into Eq. (17) and taking the limit as $\Delta \rightarrow 0$, the energy density simplifies to:

$$\begin{aligned} \lim_{\Delta \rightarrow 0} W_T &= -\frac{\rho\omega^2}{2} \lim_{\Delta \rightarrow 0} \frac{1}{1-\lambda^2} (1 - \lambda^2 - 2\lambda \text{Im}(p_1 p_2^*)(k_2 - k_1)r - \lambda(\text{Re}(p_1 p_2^*) - \lambda)(k_2 - k_1)^2 r^2) \\ &= -\frac{\rho\omega^2}{2} (1 + \alpha_1 r + \alpha_2 r^2) \end{aligned} \quad (19)$$

The expressions for the polynomial coefficients are derived as follows:

$$\begin{aligned} \alpha_1 &= -\text{Im}(\mathbf{u}_p^* \times \mathbf{u}_p) \frac{D_{EP}}{\omega^2} k_{EP}^3 \\ \alpha_2 &= \frac{(1 - p_{EP} p_{EP}^*) D_{EP}^2}{\omega^4} k_{EP}^6 \end{aligned} \quad (20)$$

It is worth noting that $\text{Im}(\mathbf{u}_p^* \times \mathbf{u}_p)$ captures the spin polarization of the excitation. The resulting energy growth near the EP exhibits a quadratic dependence on the propagation distance r , accompanied by an additional linear term arising from spin-induced asymmetry. This form of polynomial amplification is a distinctive hallmark of non-Hermitian degeneracies, standing in stark contrast to the exponential amplification characteristic of the broken-phase.

2.2.3. Case 3: Complex eigenvalues and non-orthogonal eigenvectors (Broken-phase)

When $\Delta < 0$, the system enters the broken-phase, where the eigenvalues form a pair of complex conjugates. This results in the exponential amplification or attenuation of elastic waves, depending on the propagation direction. In this regime, the kinetic energy density is given by:

$$W_T = -\frac{\rho\omega^2}{2(1 - |\lambda|^2)^2} (|p_1 - \lambda p_2|^2 e^{2\text{Im}(k_1)r} + |p_2 - \lambda^* p_1|^2 e^{2\text{Im}(k_2)r}). \quad (21)$$

This expression demonstrates that the energy can either grow or decay exponentially, depending on the imaginary parts of the complex wave numbers and the modal alignment of the excitation. Notably, when the excitation vector is fully aligned with a single eigenmode (i.e., $p_1 = 1$ or $p_2 = 1$), the system exhibits maximal amplification or attenuation, underscoring the critical role of modal polarization in energy control. This directional gain-loss behavior is a defining feature of non-Hermitian systems in the broken-phase. It is worth noting that, although the exponential growth of energy will inevitably introduce nonlinear effects, our analysis is strictly linear and restricted to infinitesimal deformations; nonlinear and finite-deformation effects, as well as the fully nonlinear stability problem, lie beyond the scope of this work. In physical systems, however, the exponential growth predicted by the linear theory will eventually drive the response into a nonlinear regime, where material and geometric nonlinearities, together with gain saturation, are expected to arrest the growth and yield a finite-amplitude steady propagation state, consistent with observations in non-Hermitian elastic metabeams with nonlinear and nonlocal coupling (Qian et al., 2025).

3. Homogenization theory for 2D odd lattices

To bridge the gap between discrete lattice microstructures and continuum-scale descriptions, we develop a homogenization theory tailored to 2D odd elastic lattices. These lattices, composed of tensile-torsional asymmetric springs, inherently exhibit non-conservative mechanical behavior due to their odd elasticity. The goal of this section is to derive effective continuum parameters that accurately capture the wave dynamics and energy-transfer characteristics observed at the microscale. By analyzing the unit-cell response and applying asymptotic and energy-based averaging techniques, we construct a closed-form representation of the effective elasticity tensor. The resulting homogenized model is then validated against full-scale discrete simulations, demonstrating its ability to predict dispersion relations, mode shapes, and direction-dependent amplification phenomena with high fidelity.

3.1. 1D odd elastic elements

We begin by examining the non-conservative behavior of the so-called “odd spring” active element, schematically illustrated in Fig. 2(a). When an axial displacement Δu is applied, the element generates both an axial force F_{s1} and a shear force F_{r1} , reflecting a mechanically coupled response (panel (i)). Conversely, when a shear displacement Δv is imposed, the element produces both a shear force F_{r2} and an axial force F_{s2} (panel (ii)). This bidirectional force-displacement response highlights the intrinsic coupling between axial and shear deformation modes, a hallmark of non-conservative behavior in odd elastic systems. This characteristic behavior can be practically realized using beam structures integrated with active control systems, such as those employing piezoelectric sensors

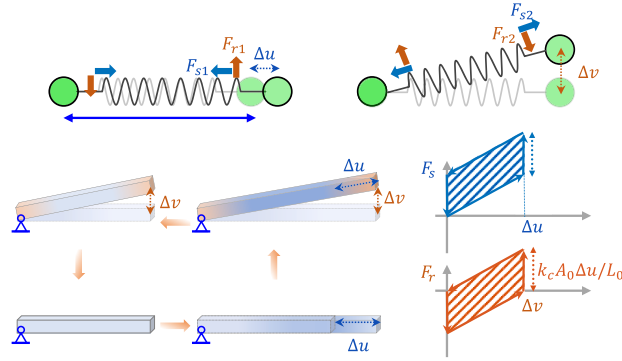


Fig. 2. Active element and its non-conservative characteristics. (a) Schematic illustration of asymmetric stretch twist coupling: (i) an axial displacement Δu induces force components F_{s1} and F_{r1} . (ii) a shear displacement Δv induces F_{s2} and F_{r2} . (b) Deformation cycle of the odd elastic element: (i) undeformed configuration. (ii) pure axial extension. (iii) combined axial and shear deformation. (iv) pure transverse deformation. (c) Force-displacement loops corresponding to shear force F_s vs. Δu and rotational force F_r vs. Δv , showing non-conservative energy exchange due to the coupling modulus k_c .

and actuators (Chen et al., 2020, 2021; Wang et al., 2024a). The mechanical behavior of the odd spring can be captured by the following constitutive relation:

$$\begin{bmatrix} k_{ss} & k_{sr} \\ k_{rs} & k_{rr} \end{bmatrix} \begin{bmatrix} \Delta u/L_b \\ \Delta v/L_b \end{bmatrix} = \begin{bmatrix} F_s/A_b \\ F_r/A_b \end{bmatrix}. \quad (22)$$

Here, k_{ss} and k_{rr} represent the axial and shear moduli, respectively, while k_{sr} and k_{rs} denote the coupling terms between the deformation modes. These moduli are defined per unit length L_b and cross-sectional area A_b of the undeformed element. In Hermitian (reciprocal) systems, the stiffness matrix is symmetric, such that $k_{sr} = k_{rs}$, as demonstrated in the reciprocal torque spring model (Craster and Guenneau, 2013). This symmetry is a fundamental feature of such systems, guaranteeing energy conservation during mode interactions. In contrast, non-Hermitian systems break this symmetry, allowing for asymmetric coupling. For simplicity, we focus on the fully antisymmetric case in which $k_{sr} = -k_{rs} = k_c$, where k_c characterizes the odd elasticity coefficient.

To investigate the energy behavior of the system, we consider the closed deformation loop illustrated in Fig. 2(b). The deformation proceeds through the following four-step sequence: (i) the initial undeformed state, (ii) pure axial extension, (iii) simultaneous axial and shear deformation, (iv) pure shear deformation, after which the system returns to its original configuration. Throughout this closed deformation cycle, the change in the system's elastic potential energy is given by:

$$W_e = \oint F_s du + \oint F_r dv = 2k_c A_0 \Delta u \Delta v / L_0. \quad (23)$$

The two integrals represent the mechanical work performed by the axial and shear forces, respectively. As shown in Fig. 2(c), the force-displacement curves clearly illustrate the non-conservative nature of the system: the shaded area enclosed by the deformation cycle represents the net energy gain or loss intrinsic to the odd spring.

From a micromechanical perspective, the overall behavior of a lattice system is dictated by the constitutive properties of its individual elements. As a result, materials assembled from odd spring components inherently exhibit non-conservative behavior at the continuum scale. This insight forms the basis for the subsequent sections, in which we introduce a systematic design methodology and a homogenization framework for realizing bulk odd elastic media. These approaches establish a direct connection between microstructural architecture and macroscopic mechanical response, laying the groundwork for the practical implementation of non-Hermitian mechanical systems.

3.2. Homogenization of 2D odd lattices with diverse microstructural designs

Odd mechanical lattices constructed from active elements represent a novel class of architected materials, purposefully engineered at the microscale to introduce non-conservative energy interactions. This design paradigm enables mechanical behaviors that transcend the constraints of classical (conservative) elasticity. Such lattices, whether based on square, triangular, or modified hexagonal configurations, are composed of periodically repeated unit cells formed by networks of active elements connected at discrete nodal junctions. These unit cells are arranged along independent lattice vectors to create translationally invariant structures. Representative examples of these classical lattice geometries are shown in Fig. 3.

Owing to the intrinsically non-conservative nature of their constituent elements, these lattices exhibit non-conservative behavior at the macroscopic scale. To quantitatively characterize this response, we adopt a multiscale homogenization framework that systematically derives the effective continuum properties of the system. This approach establishes a rigorous link between the microscale architecture and the emergent macroscale mechanical behavior.

As illustrated in Fig. 3, the positions of the dependent nodes (orange) can be expressed in terms of the independent nodes (blue) along the lattice vectors $\mathbf{a}_1$ and $\mathbf{a}_2$, where L_0 denotes the lattice constant. Under the application of a uniform macroscopic strain

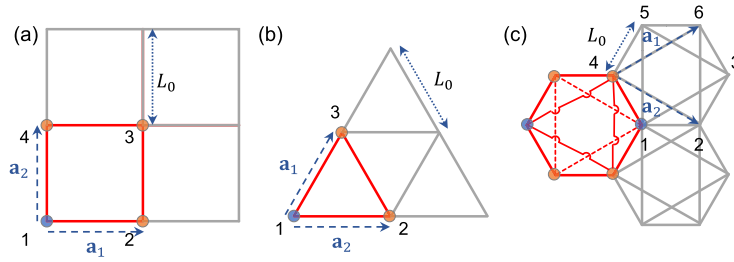


Fig. 3. 2D odd lattices with various microstructure designs. (a) Square lattice; (b) triangular lattice; (c) modified hexagonal lattice incorporating long-range odd elastic spring elements in its interior.

field, the lattice deforms coherently. In this study, macroscale to microscale transition relation is formulated in terms of the full displacement gradient rather than the symmetric Cauchy strain. Such a description requires a generalized Cauchy-Born construction (Erickson, 2008; Tadmor et al., 1996), and the detailed derivation is presented in Appendix B.1. According to the hypothesis, the displacements of the lattice nodes, such as $\mathbf{u}_i$ (where the subscript denotes the node index), are linearly related to the components of the macroscopic strain tensor and can be expressed as:

$$\begin{aligned} \mathbf{u}_1 &= \mathbf{u}_1 \\ \mathbf{u}_2 &= \mathbf{u}_1 + \boldsymbol{\varepsilon} \cdot \mathbf{a}_1 \\ \mathbf{u}_3 &= \mathbf{u}_1 + \boldsymbol{\varepsilon} \cdot (\mathbf{a}_1 + \mathbf{a}_2) \\ \mathbf{u}_4 &= \mathbf{u}_1 + \boldsymbol{\varepsilon} \cdot \mathbf{a}_2 \end{aligned} \quad (24)$$

These equations can be equivalently reformulated in matrix form, providing a more compact representation that facilitates subsequent analytical and computational treatment:

$$\mathbf{d} = \mathbf{B}_0 \cdot \mathbf{d}_0 + \mathbf{B}_e \cdot \boldsymbol{\varepsilon}_{\text{mac}}, \quad (25)$$

where $\boldsymbol{\varepsilon}_{\text{mac}} = [\varepsilon_{11}, \varepsilon_{22}, \varepsilon_{12}, \varepsilon_{21}]^T$ denotes the macroscopic odd strain vector. The vectors $\mathbf{d}$ and $\mathbf{d}_0$ represent the displacement vectors of all lattice nodes and the subset of independent nodes, respectively. The matrices $\mathbf{B}_0$ and $\mathbf{B}_e$ are coefficient matrices constructed based on the lattice geometry and nodal connectivity, mapping the macroscopic strain to nodal displacements. By systematically aggregating the force-displacement relationships of the individual odd springs, the global stiffness matrix $\mathbf{K}_g$ is assembled, encapsulating the overall mechanical behavior of the unit cell. This matrix establishes a linear relationship between the nodal displacement vector $\mathbf{d}$ and the corresponding nodal force vector $\mathbf{f}$, expressed as:

$$\mathbf{K}_g \cdot \mathbf{d} = \mathbf{f}. \quad (26)$$

The global stiffness matrix $\mathbf{K}_g$ is assembled from the microscopic stiffness relations (cf. Eq. (22)). Due to the intrinsically non-conservative nature of the system, the effective elasticity tensor generally cannot be derived as the derivative of a scalar strain-energy function, even though path-dependent work can still be consistently evaluated and compared. Instead, the macroscopic odd stress tensor is computed via a dyadic product between the position vectors and their corresponding nodal force vectors, expressed as:

$$\boldsymbol{\sigma}_{\text{mac}} = \frac{1}{V_{\text{Area}}} \sum_{i=1}^4 \mathbf{x}_i \otimes \mathbf{F}_i. \quad (27)$$

The area of the unit cell is defined as $V_{\text{Area}} = L_0^2$, where $\mathbf{x}_i$ and $\mathbf{F}_i$ denote the position and nodal force vectors of the i th node, respectively. By applying a series of prescribed macroscopic odd strain fields and computing the resulting stress responses, the effective elasticity tensor $\mathbf{K}$ can be systematically extracted through an appropriate post-processing procedure, as outlined below:

$$\boldsymbol{\sigma}_{\text{mac}} = \mathbf{K} \cdot \boldsymbol{\varepsilon}_{\text{mac}}. \quad (28)$$

Accordingly, for a square lattice composed of odd springs, the effective elasticity matrix can be explicitly formulated as:

$$\mathbf{K}_{\text{eff}} = 2 \begin{bmatrix} k_{ss} & 0 & k_c & 0 \\ 0 & k_{ss} & 0 & -k_c \\ -k_c & 0 & k_{rr} & 0 \\ 0 & k_c & 0 & k_{rr} \end{bmatrix}. \quad (29)$$

Here, k_{ss} , k_{rr} , and k_c denote the axial, shear, and coupling modulus coefficients of the odd springs, respectively. Additionally, assuming that each lattice node has a mass m_0 , the effective density of the lattice is given by $\rho_{\text{eff}} = 4m_0/L_0^2$. It is important to note that the inclusion of odd springs modifies the constitutive relation such that a rigid-body rotation results in a nonzero stress response. A detailed analysis of this effect and its underlying mechanism is provided in Appendix B.2. For completeness, the effective elasticity matrices for the two additional odd-lattice configurations, the triangular lattice and the modified hexagonal lattice, are provided in Appendix C.

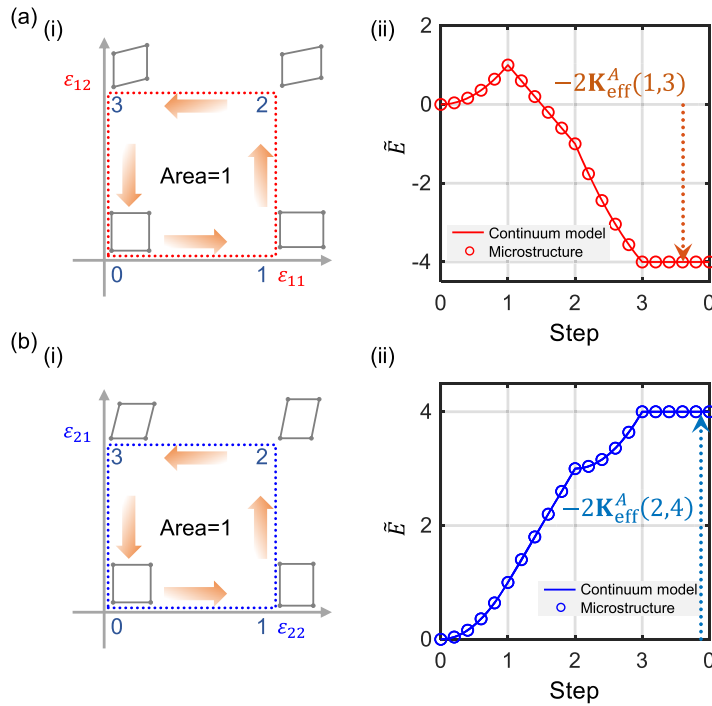


Fig. 4. Energy cycling in odd elastic lattices. Synthetic deformation cycles are applied in two strain subspaces: (a) ε_{11} - ε_{12} (Path 1) and (b) ε_{22} - ε_{21} (Path 2), each forming a closed loop of unit area in strain space (panels (i)). The corresponding evolution of homogenized energy (panels (ii)) reveals non-conservative mechanical response. Results from microstructural simulations (circles) show excellent agreement with the continuum model (solid lines).

3.3. Energy evolution over a closed strain cycle

Building on the previous analysis, the antisymmetric component of the effective elasticity matrix encapsulates the non-conservative potential responsible for energy circulation within the strain space. For clarity and without loss of generality, we focus on the square odd lattice as a representative example. To validate the theoretical predictions, we quantify the work accumulated by the lattice over prescribed deformation cycles and examine the structure of its effective elasticity matrix within the synthetic strain space:

$$\mathbf{K}_{\text{eff}}^A = (\mathbf{K}_{\text{eff}} - \mathbf{K}_{\text{eff}}^T)/2 = 2 \begin{bmatrix} 0 & 0 & k_c & 0 \\ 0 & 0 & 0 & -k_c \\ -k_c & 0 & 0 & 0 \\ 0 & k_c & 0 & 0 \end{bmatrix}. \quad (30)$$

This formulation highlights the non-conservative characteristics of the active medium when subjected to synthetic strain paths. To further illustrate this behavior, Fig. 4(a), (b)–(i) present two representative closed loops in strain space, each corresponding to specific off-diagonal components of the effective elasticity matrix defined in Eq. (30). The resulting variations in microstructural energy along these deformation paths are quantified as follows:

$$E_{\text{Micro}} = \frac{1}{V_{\text{Area}}} \oint_{\ell} \mathbf{f}^T \cdot d\mathbf{d} = \frac{1}{V_{\text{Area}}} \oint_{\ell} d^T \cdot \mathbf{K}_s^T \cdot d\mathbf{d}. \quad (31)$$

The strain energy at the continuum scale is evaluated via Eq. (4). Without loss of generality, a dimensionless energy metric is defined as $\tilde{E} = E_{\text{Homo}}(\text{or } E_{\text{Micro}})/k_c$. As illustrated in Fig. 4(a), (b)–(ii), the strain energy evolution computed from microstructural simulations closely matches that predicted by the continuum model, confirming the validity of the effective elasticity formulation. This strong consistency not only validates the homogenization approach but also reinforces the robustness of the underlying theoretical framework.

3.4. Validation of the homogenization model and dynamic characterization

To confirm the predictive accuracy of the homogenized model for odd elastic lattices, rigorous validation against full-scale microstructural simulations is essential. While the static response of the odd medium was addressed in the preceding subsection, dynamic validation, particularly in regimes where odd elastic effects are pronounced, is equally critical. Key aspects of this validation include the analysis of dispersion relations and wave propagation characteristics, with particular emphasis on the emergence of non-reciprocal

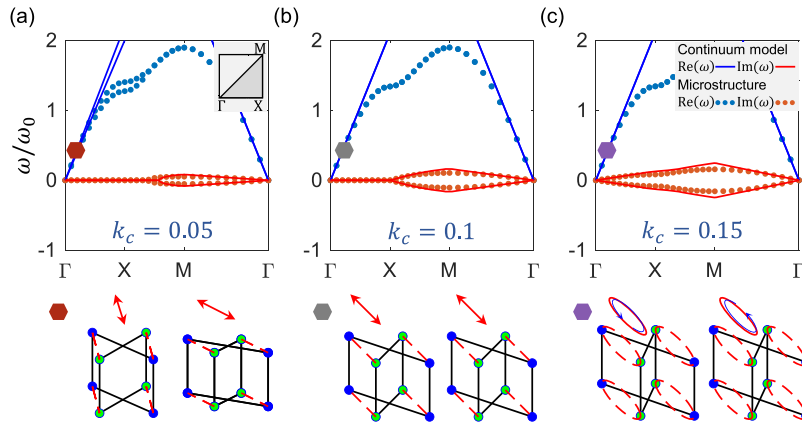


Fig. 5. Dynamic properties under varying coupling coefficients. (a-c) Dispersion relations obtained from the continuum model (solid lines) and microstructural simulations (dots), shown for three representative values of the coupling coefficient: $k_c = 0.05$, 0.1 , and 0.15 , respectively. The real and imaginary parts of the eigenfrequencies are plotted in blue and red, respectively. Lower panels depict the corresponding microstructural mode shapes at the highlighted points (hexagonal markers), with red arrows indicating polarization directions predicted by the effective medium.

transport phenomena. Additionally, multiphase dynamic analysis is necessary to capture the complex interactions between different wave regimes. As discussed previously, these regimes are classified according to the discriminant $\Delta = (D_{11}^S - D_{22}^S)^2 + 4(D_{12}^S)^2 - D_{12}^A^2$, which divides the system into three distinct phases: the unbroken-phase, the EP, and the broken-phase. Each phase is characterized by unique wave propagation and energy transport behaviors, providing a comprehensive framework for assessing the validity and robustness of the homogenized model. For the case of the odd square lattice, this criterion simplifies along the x -direction to: $\Delta = ((k_{ss} - k_{rr})^2 - 4k_c^2)/(4m_0^2)$. In the following analysis, we fix the modulus parameters as $k_{ss} = 1$ and $k_{rr} = 0.8$, and investigate the dispersion behavior for various values of the coupling coefficient k_c . As shown in Fig. 5(a)–(c), excellent agreement is observed between the continuum and microstructural models in both the real and imaginary parts of the dispersion relation, particularly in the long-wavelength regime, when normalized by $\omega_0^2 = k_{ss}/m_0$. This result confirms the effectiveness of odd elasticity as a continuum framework for modeling the dynamics of non-Hermitian lattices.

Furthermore, along the x -direction in the Brillouin zone, the evolution of the dispersion with increasing coupling coefficient k_c can be viewed as a phase transition: from purely real dispersion relations characteristic of the unbroken-phase (Fig. 5(a)), to mode coalescence at the exceptional point (Fig. 5(b)), and ultimately to the emergence of complex-conjugate dispersion branches in the broken-phase (Fig. 5(c)). The associated mode shapes, presented in the lower panels and indicated by hexagons, visualize polarization orientations using red arrows, providing deeper insight into the evolution of the system's modal structure across these regimes. Overall, these results validate the capability of the homogenized model to accurately capture both dispersion behavior and modal polarization, reinforcing the robustness of the proposed continuum framework in characterizing the dynamic response of odd elastic systems.

To further elucidate the anomalous wave amplification phenomena observed in non-Hermitian systems under varying conditions, we consider a one-dimensional array composed of square lattice unit cells to demonstrate its capability for elastic wave energy manipulation. For comparison, wave propagation in a homogeneous odd elastic medium is also simulated by modifying the weak-form PDE module in COMSOL. The simulation results are presented in Fig. 6, where elastic waves are incident from the left side of the waveguide. Perfectly matched layers (PMLs) are applied at both ends to effectively suppress reflections and emulate open-boundary conditions.

Complex band structure evolution and energy flux analyses provide comprehensive insight into how distinct spectral phases give rise to emergent non-Hermitian dynamics. Together, these results not only validate the homogenized continuum model but also establish a robust foundation for engineering tunable wave amplification in elastic media. As illustrated in Fig. 6(a)–(c), the energy amplification behavior varies significantly across different non-Hermitian regimes, in close agreement with theoretical predictions. In the unbroken-phase regime (Fig. 6(a)), elastic waves exhibit sinusoidal amplification driven by a small imaginary component in the dispersion relation, indicative of weak gain without symmetry breaking. As the system approaches the exceptional point (Fig. 6(b)), wave amplitudes grow quadratically, reflecting the coalescence of eigenmodes and the critical spectral topology of the EP. In the broken-phase regime (Fig. 6(c)), the emergence of complex-conjugate eigenvalues with dominant imaginary parts results in exponential energy amplification during propagation. Both continuum and microstructural simulations consistently capture these distinct amplification characteristics, as shown in Fig. 6(d), with their quantitative agreement further confirmed in the same figure. This cross-scale consistency reinforces the validity, robustness, and practical relevance of the proposed odd elasticity framework in describing non-Hermitian elastic wave phenomena.

In an odd elastic medium, energy exhibits non-conservative behavior during cyclic deformation in strain space. Analogously, the vibration of a material element under elastic wave propagation can be interpreted as a form of cyclic deformation, typically

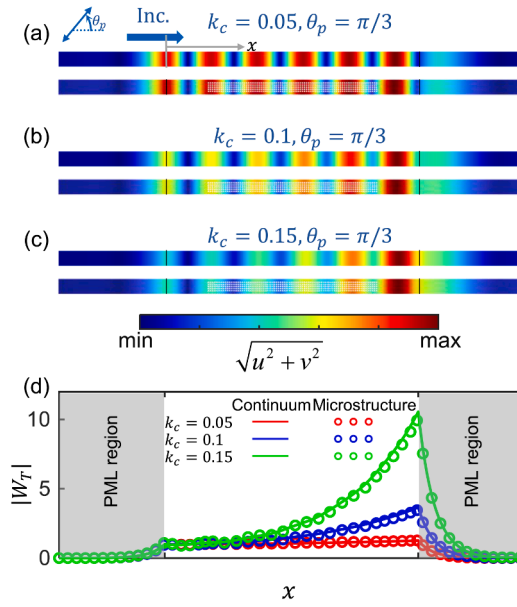


Fig. 6. Elastic wave propagation in non-Hermitian medium under different coupling coefficients k_c , with a fixed incident angle $\theta_p = \pi/3$. (a-c) correspond to increasing values of k_c : 0.05 (unbroken-phase), 0.1 (EP), and 0.15 (broken-phase), respectively. (d) Kinetic energy density along the propagation direction, comparing results from the continuum model and the microstructural simulation.

represented by the displacement field: $\mathbf{u}(t) = \mathbf{u}_0(t)e^{-i\mathbf{k}\cdot\mathbf{r}+i\omega t}$. Based on this formulation, the corresponding energy variation over a single oscillation period, $T_0 = 2\pi/\omega$, can be evaluated as:

$$\Delta W = \int_0^{T_0} \mathbf{u}^\dagger(t) \cdot \mathbf{u}(t) dt. \quad (32)$$

To further investigate the energy evolution associated with polarization in odd elastic media, we consider the case $\omega/\omega_0 = 0.1$. The work done by external forces over one vibration cycle is computed for both a homogeneous medium and a structured lattice, with the results presented in Fig. 7. In the Hermitian case (Fig. 7(a)), the external forces perform no net work during the cycle, consistent with energy conservation. In contrast, for non-Hermitian systems (Fig. 7(b)–(d)), increasing the coupling coefficient leads to progressively larger negative work, reflecting net energy gain within the medium. This trend corresponds to a transition in the potential energy response, from sinusoidal amplification in the unbroken-phase, to quadratic growth near the EP, and ultimately to exponential amplification in the broken-phase. These results offer indirect yet compelling evidence supporting the proposed theoretical framework, and underscore its effectiveness in capturing the non-conservative energy dynamics induced by polarization in non-Hermitian elastic systems.

3.5. Directional propagation of elastic waves in the 2D odd lattice

Building on the previous analysis of elastic wave energy amplification in non-Hermitian square lattices, we now explore their potential for directional control of wave propagation in a 2D lattice assembled from square unit cells. At the normalized frequency $\omega/\omega_0 = 0.25$, isofrequency contours for varying odd coupling strengths are presented in Fig. 8. Subfigure (a) shows the real parts of the dispersion relations, while subfigure (b) illustrates their imaginary counterparts. Panels (i) through (iv) correspond to increasing values of the coupling coefficient k_c . Solid lines represent predictions from the continuum model, and circular markers indicate numerical results from the discrete microstructural simulations.

As discussed in the previous section, the imaginary part of the isofrequency contours captures key signatures of spatial amplification, including exponential energy growth in the broken-phase regime and quadratic amplification at the EPs. The homogenized continuum model and microstructural simulations show excellent agreement in the long-wavelength limit, thereby validating the proposed homogenization framework. The isofrequency contours reveal four distinct wave phenomena as the coupling coefficient k_c increases:

- **Case (i):** In the Hermitian regime ($k_c = 0$), the isofrequency contours are real valued and anisotropic, indicating direction-dependent wavelengths but no amplification. Wave propagation remains delocalized and energy conserving.
- **Case (ii):** In the unbroken-phase regime ($k_c = 0.05$), complex-valued dispersion relations emerge along the diagonals. Exponential amplification develops along the diagonal directions, while waves propagating along other directions remain weak or attenuated.

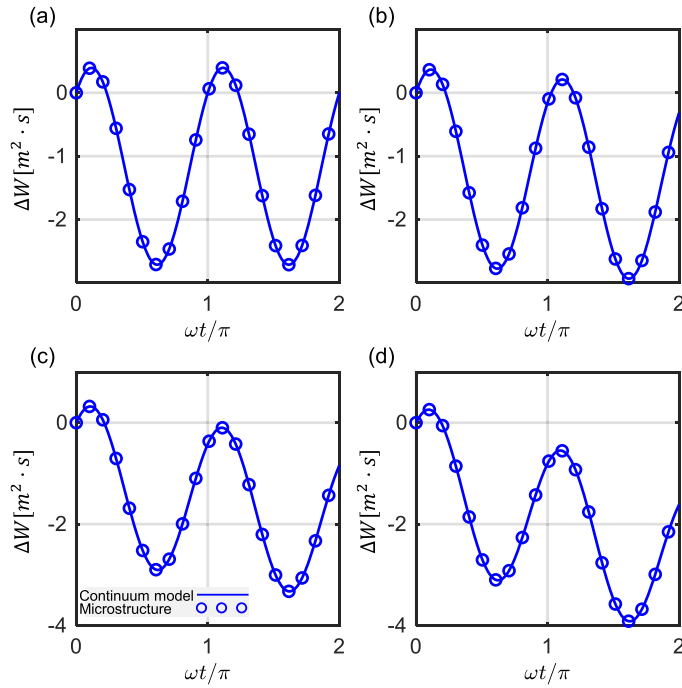


Fig. 7. Work done by external forces over one vibration cycle for varying coupling coefficients k_c , at a fixed polarization angle $\theta_p = \pi/3$ and normalized frequency $\omega/\omega_0 = 0.1$. (a) Hermitian case with $k_c = 0$; (b-d) non-Hermitian cases with increasing coupling strength, corresponding to $k_c = 0.05$, $k_c = 0.1$ (EP), and $k_c = 0.15$, respectively. Solid lines indicate results from the continuum model, while circular markers denote data from microstructural simulations.

- **Case (iii):** At the EP regime ($k_c = 0.1$), the real parts of the dispersion branches coalesce along the principal axes, producing a hybrid regime: exponential amplification persists along the diagonals, while quadratic amplification arises in the horizontal and vertical directions, a hallmark of EP-induced mode degeneracy.
- **Case (iv):** In the broken-phase regime ($k_c = 0.15$), the dispersion relations become complex-valued in all directions, leading to uniform exponential amplification across the angular spectrum. Direction-dependent amplification rates persist, yielding strong anisotropy and directional selectivity.

To further validate these spectral behaviors, full-wave simulations were performed on a 40×40 odd elastic lattice with a central monopole source operating at the same normalized frequency $\omega/\omega_0 = 0.25$. To isolate the intrinsic wave dynamics, the surrounding domain was modeled using the homogenized elasticity matrix, and PMLs were implemented at the boundaries to eliminate spurious reflections. Fig. 8(c) presents the displacement fields for the four representative values of the odd coupling coefficients k_c , with panels (i)-(iv) directly corresponding to the spectral regimes discussed in cases (i)-(iv) below. The observed field patterns exhibit excellent agreement with the predictions derived from the isofrequency contours, confirming the effectiveness of the spectral framework in capturing directional amplification and localization phenomena.

When odd coupling is absent (case i), the system reduces to a conventional Cauchy medium. The simulated displacement field remains largely delocalized and shows no evidence of amplification, consistent with the purely real, anisotropic isofrequency contours that characterize Hermitian systems. As odd coupling is introduced (case ii), the wave response changes markedly. The displacement amplitude grows exponentially along the lattice diagonals, while wave propagation in other directions is significantly suppressed. This strong anisotropic amplification arises from the directional gain encoded in the imaginary parts of the now complex-valued isofrequency contours. At the EP (case iii), where two eigenmodes coalesce, the system exhibits a hybrid amplification behavior. While exponential growth along the diagonals persists, it is now accompanied by weaker, quadratic amplification along the principal axes. This mixed response reflects the critical spectral topology of the EP and signifies the transition between the unbroken and broken-phases. Once the system enters the broken-phase regime (case iv), the isofrequency contours become fully complex, giving rise to stronger and more broadly distributed directional amplification. The resulting displacement fields reveal significant energy accumulation along multiple directions, consistent with the extended gain structure predicted by the spectral analysis.

Collectively, these results demonstrate that spatial wave amplification in odd elastic media can be precisely modulated via the odd coupling coefficient k_c , with clear correspondence between spectral geometry and field behavior. This tunability enables programmable control over both the magnitude and directionality of wave amplification, while also facilitating smooth transitions between quadratic and exponential gain regimes. It is important to emphasize that, although conventional square lattices can exhibit directional energy transport as a consequence of elastic anisotropy, this behavior is purely conservative and does not involve directional gain. By contrast, in odd elasticity, the imaginary part of the dispersion relation endows specific propagation directions with net

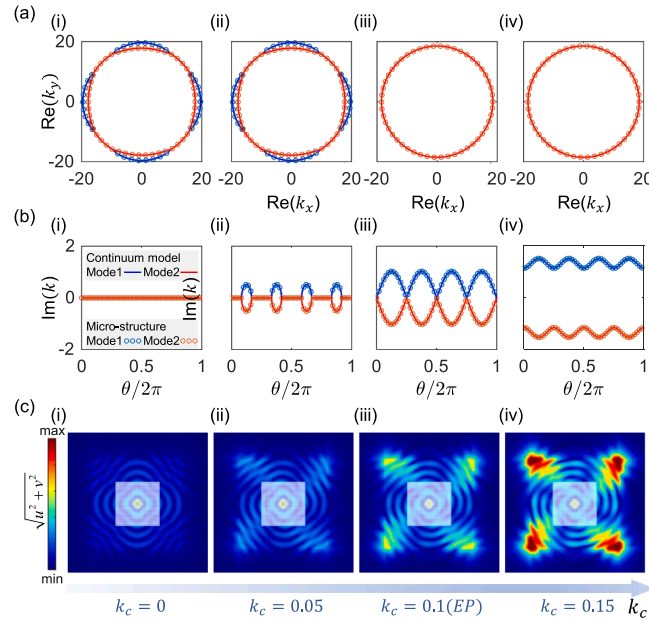


Fig. 8. Comparison of isofrequency contours and displacement fields between the continuum and microstructured models. (a) Real and (b) imaginary parts of the isofrequency contours at a normalized frequency $\omega/\omega_0 = 0.25$, for four representative values of the coupling coefficient: (i) $k_c = 0$ (Hermitian regime), (ii) $k_c = 0.05$ (unbroken-phase regime), (iii) $k_c = 0.1$ (EP), and (iv) $k_c = 0.15$ (broken-phase regime). (c) Displacement field contours for the corresponding cases (i-iv), illustrating the evolution of spatial amplification and directionality under increasing non-Hermitian coupling.

amplification or attenuation, so that directional transport is intrinsically tied to non-Hermitian energy growth, a feature that cannot be realized in classical Hermitian lattices. Additional strategies for tailoring amplification along alternative propagation directions are discussed in [Appendix C](#). Such flexibility lays a robust foundation for designing active, non-reciprocal, and direction-sensitive elastic metamaterials with highly customizable dynamic functionalities.

4. Unprecedented wave manipulation in 2D odd media

2D odd elastic media introduce a new paradigm for wave manipulation, enabled by their inherently non-Hermitian and non-reciprocal mechanical properties. In contrast to conventional elastic materials, where wave propagation is governed by symmetric constitutive relation, odd media incorporate antisymmetric elastic moduli, thereby enabling localized and directional energy injection or dissipation. This unique capability gives rise to a range of unconventional wave phenomena, including unidirectional wave propagation, asymmetric mode conversion, and direction-dependent amplification or attenuation. In this section, we explore how 2D odd elastic media can facilitate non-reciprocal surface waves and polarization-dependent bulk mode conversion, demonstrating their vast potential for the next generation of mechanical wave control.

4.1. Non-reciprocal surface wave propagation

Surface waves, particularly Rayleigh waves, propagate along material interfaces while decaying exponentially into the bulk ([Rayleigh, 1885](#)), and play a important role in applications ranging from geophysical exploration to engineered waveguides. In odd elastic medium, these waves are significantly influenced by non-Hermitian effects arising from non-reciprocal couplings and geometric asymmetries. Building upon this foundation, this subsection investigates the control of surface wave propagation in non-Hermitian elastic lattices. Specifically, we analyze the dispersion characteristics of surface waves and demonstrate how non-reciprocal interactions and structural asymmetry modulate their directional behavior and localization.

As illustrated in [Fig. 9\(a\)](#), we examine surface waves propagating along a horizontal interface, with the associated displacement field assumed to follow the form:

$$\mathbf{u}^r = \mathbf{u}^r e^{-ik_x(x+\xi y)+i\omega t}. \quad (33)$$

Here, k_x denotes the wave vector in the direction of propagation, and ξ represents the decay coefficient describing the exponential attenuation of the wave into the bulk. The phase velocity of the surface wave is given by $c_r = \omega/k_x$. For rightward (positive) propagation, the conditions $k_x > 0$ and $\text{Im}(\xi) > 0$ ensure physically admissible decay. Conversely, for leftward (negative) propagation, $k_x < 0$ and $\text{Im}(\xi) < 0$ must hold. The existence of surface waves requires that two conditions be satisfied simultaneously: (i) the bulk wave

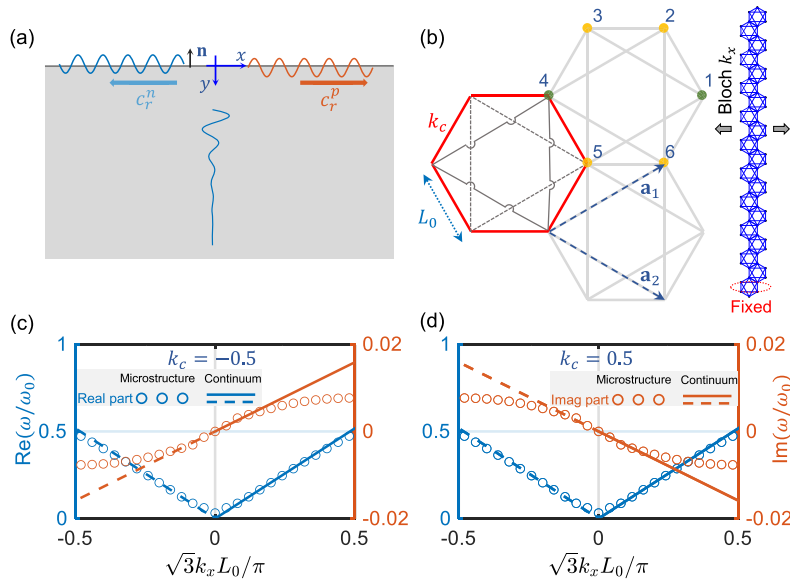


Fig. 9. Complex dispersion of surface waves in the modified hexagonal lattice. (a) Schematic of a surface wave propagating along the x -direction with exponential decay into the bulk. (b) Modified hexagonal lattice composed of conventional springs (black) and odd springs (red) with coupling modulus k_c ; right: corresponding strip model with Bloch-periodic boundary conditions along the x -direction. (c, d) Real and imaginary parts of the surface wave dispersion for $k_c = \mp 0.5$. Solid and dashed lines represent the real and imaginary frequency components predicted by the continuum model, respectively, while circular markers denote results from microstructural simulations.

equation must admit a decaying solution, and (ii) the stress-free boundary condition must be enforced at the surface. These conditions are expressed as:

$$\begin{cases} \nabla \cdot \boldsymbol{\sigma}^r = -\omega^2 \rho \mathbf{u}^r \\ \mathbf{t} = \mathbf{n} \cdot \boldsymbol{\sigma}^r|_{y=0} = \mathbf{0} \end{cases} \quad (34)$$

Let $\mathbf{n}$ denote the unit normal vector to the interface. In the Cartesian coordinate system, the component-wise form of Eq. (34) can be expressed as:

$$\begin{cases} \{(K_{1m1n} - \rho c_r^2 \delta_{mn}) + \xi(K_{1m2n} + K_{2m1n}) + \xi^2 K_{2m2n}\} u_n^r = 0 \\ t_m = -ik_x(K_{2m1n} + \xi K_{2m2n}) u_n^r = -ik_x I_m = 0 \end{cases} \quad (35)$$

Here, the subscripts m and n refer to the tensor components, and $\mathbf{I}$ denotes the stress polarization vector associated with the surface wave. For analytical convenience, we define the following intermediate variables:

$$\begin{aligned} Q_{mn} &= K_{1m1n} - \rho c_r^2 \delta_{mn} \\ R_{1mn} &= K_{1m2n} \\ R_{2mn} &= K_{2m1n} \\ T_{mn} &= K_{2m2n} \end{aligned} \quad (36)$$

Accordingly, Eq. (35) can be further expressed in matrix form as:

$$\begin{cases} (\mathbf{Q} + \xi(\mathbf{R}_1 + \mathbf{R}_2) + \xi^2 \mathbf{T}) \cdot \mathbf{u}^r = \mathbf{0} \\ \mathbf{I} = (\mathbf{R}_2 + \xi \mathbf{T}) \cdot \mathbf{u}^r \end{cases} \quad (37)$$

The first equation in Eq. (37) constitutes a generalized quadratic eigenvalue problem with respect to the eigenvalue ξ . Due to its nonlinear nature, obtaining a closed-form analytical solution is generally intractable. To address this challenge, the Stroh formalism (Stroh, 1962) introduces the stress polarization vector as part of the eigenvector, thereby reformulating Eq. (37) into a linear eigenvalue problem. Consequently, the system can be expressed as an eigenvalue problem involving both the displacement polarization vector $\mathbf{u}_0$ and the stress polarization vector $\mathbf{I}$:

$$\begin{bmatrix} -\mathbf{T}^{-1} \cdot \mathbf{R}_2 & \mathbf{T}^{-1} \\ -\mathbf{Q} + \mathbf{R} \cdot \mathbf{T}^{-1} \cdot \mathbf{R}_2 & -\mathbf{R} \cdot \mathbf{T}^{-1} \end{bmatrix} \begin{bmatrix} \mathbf{u}^r \\ \mathbf{I} \end{bmatrix} = \xi \begin{bmatrix} \mathbf{u}^r \\ \mathbf{I} \end{bmatrix}. \quad (38)$$

The phase velocities generally admit two distinct solutions, denoted as c_r^p and c_r^n , corresponding to wave propagation in the positive and negative x -directions, respectively. In classical Cauchy media, time-reversal symmetry imposes the constraint $c_r^p + c_r^n = 0$, indicating that the wave speeds are equal in magnitude but opposite in direction. However, this symmetry is broken in odd elastic media, and the relation no longer holds.

The stress polarization vectors associated with surface waves propagating in opposite directions are denoted as $\mathbf{I}_{p1}$, $\mathbf{I}_{p2}$, $\mathbf{I}_{n1}$, and $\mathbf{I}_{n2}$, where the subscripts 1 and 2 refer to the two polarization modes of the elastic wave, and the letters p and n indicate propagation in the positive and negative x -directions, respectively. Accordingly, the surface traction vector $\mathbf{t}$ can be represented as a linear combination of these stress polarization vectors:

$$\mathbf{t} = \begin{cases} -ik_x(c_1\mathbf{I}_{p1} + c_2\mathbf{I}_{p2}), k_x > 0 \\ -ik_x(c_1\mathbf{I}_{n1} + c_2\mathbf{I}_{n2}), k_x < 0 \end{cases} \quad (39)$$

Where c_1 and c_2 are arbitrary coefficients. By enforcing the stress-free boundary condition at the surface, the following constraint must be satisfied:

$$\begin{cases} \mathbf{I}_{p1}c_1 + \mathbf{I}_{p2}c_2 = \mathbf{0}, k_x > 0 \\ \mathbf{I}_{n1}c_1 + \mathbf{I}_{n2}c_2 = \mathbf{0}, k_x < 0 \end{cases} \quad (40)$$

A nontrivial solution to this equation exists if and only if the determinant of the coefficient matrix vanishes, i.e.,

$$\begin{cases} \text{Det}[\mathbf{I}_{p1} \quad \mathbf{I}_{p2}] = 0, k_x > 0 \\ \text{Det}[\mathbf{I}_{n1} \quad \mathbf{I}_{n2}] = 0, k_x < 0 \end{cases} \quad (41)$$

Based on the above equations, the complex phase velocities $c_{n,p}^r$ corresponding to wave propagation in the negative and positive x -directions respectively, can be directly obtained.

To demonstrate the design methodology for surface waves, we consider a modified hexagonal lattice with lattice constant L_0 , as illustrated in Fig. 9(b). In this configuration, black lines represent conventional (Hermitian) springs, while red lines indicate odd springs characterized by a coupling modulus k_c . Using the homogenization framework introduced earlier, the effective elasticity tensor for this lattice can be expressed as:

$$\mathbf{K}_{\text{eff}} = \frac{1}{2\sqrt{3}} \begin{bmatrix} 11k_{ss} & 5k_{ss} & 0 & 0 \\ 5k_{ss} & 11k_{ss} & 0 & 0 \\ 2k_c & 2k_c & 3k_{ss} & 3k_{ss} \\ -2k_c & -2k_c & 3k_{ss} & 3k_{ss} \end{bmatrix}. \quad (42)$$

Based on the effective elasticity matrix, the corresponding complex dispersion relation for surface waves can be derived. For validation, we also consider a strip composed of 20 unit cells with Bloch-periodic boundary conditions imposed along the left and right edges, as illustrated in Fig. 9(b). The complex dispersion of this microstructured system is calculated in a similar manner. The results are compared in Fig. 9(c), (d), where the solid and dashed lines correspond to the dispersion branches of surface waves propagating in the positive and negative directions along the interface, respectively, and circular markers indicate results from the discrete microstructural model. Excellent agreement is observed between the continuum and discrete approaches in the long-wavelength limit.

Moreover, for negative coupling modulus ($k_c < 0$), the surface wave energy amplifies along the negative propagation direction while decaying along the positive direction, resulting in exponential energy localization at the left boundary. Conversely, for positive coupling ($k_c > 0$), energy decays in the negative direction and amplifies in the positive direction, leading to localization at the right boundary. This asymmetric amplification and attenuation behavior, in which wave energy grows exponentially in one direction and decays in the opposite, is known as the non-Hermitian skin effect (Okuma et al., 2020; Cheng and Hu, 2022). In our formulation, it is characterized by a pair of complex-conjugate phase velocities that possess identical imaginary parts but opposite real parts. Under specific boundary conditions, this feature results in exponential energy localization at one edge of the system, which constitutes an intrinsic manifestation of non-Hermitian topology.

The skin effect can be further interpreted from a topological perspective. As depicted in Fig. 10(a), the complex dispersion curves in (k_x, ω) space are projected onto the complex frequency plane ($\text{Re}(\omega)$, $\text{Im}(\omega)$), with arrows indicating the direction of increasing wavevector k_x . As the coupling coefficient k_c varies from -0.5 to 0.5 , the closed trajectory of the band structure undergoes a reversal in orientation, which serves as an indicator of band inversion. This topological transition is quantified by the non-Hermitian winding number v_w (Zhang et al., 2022), a topological invariant that counts the number of times the frequency trajectory winds counterclockwise around a reference point ω_b in the complex plane. It is defined as Duren et al. (1996):

$$v(\omega_b) = \frac{1}{2\pi i} \oint_{\text{IBZ}} \frac{d}{dk} \log(\omega(k) - \omega_b) dk. \quad (43)$$

Here, wave vectors are evaluated along the boundary of the IBZ, with the integral taken over a closed loop in k -space. After simplification, the winding number reduces to $v_w(\omega_b) = \text{sgn}(k_c)$, indicating energy localization on the negative side for $v_w = -1$ and on the positive side for $v_w = 1$. This directional dependence aligns with the complex phase velocity analysis presented earlier.

To further validate the theoretical predictions, we examine a homogeneous odd elastic medium, as illustrated in Fig. 10(b), (c). The top surface is traction-free, while the remaining boundaries are enclosed by PMLs of finite thickness to suppress wave reflections. A harmonic displacement excitation at frequency ω_b is applied at the midpoint of the free surface, oriented perpendicular to the boundary. Simulation results from the continuum model, shown in Fig. 10(b), reveal that surface wave energy propagates predominantly in one direction and undergoes exponential amplification, ultimately localizing near one edge of the system. The direction of this energy localization is governed by the sign of the odd spring modulus k_c : when $k_c < 0$, energy accumulates at the left boundary; when $k_c > 0$, it concentrates at the right boundary. This unidirectional energy amplification and spatial confinement are hallmark features of the non-Hermitian skin effect, and the results are in excellent agreement with theoretical predictions derived from

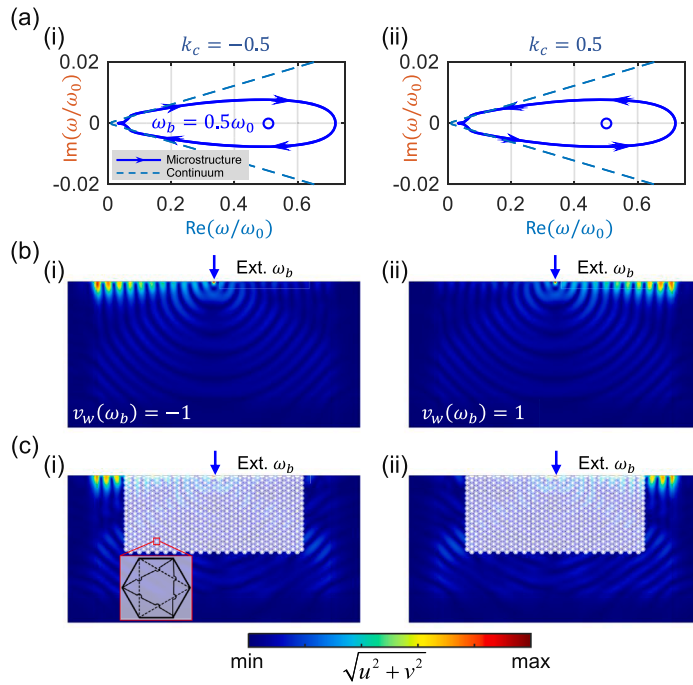


Fig. 10. Topological characterization and simulation of the surface wave skin effect. (a) Complex frequency trajectories in the $(\text{Re}(\omega), \text{Im}(\omega))$ plane mapped from the Brillouin zone for (i) $k_c = -0.5$ and (ii) $k_c = 0.5$. Arrows indicate the direction of increasing wavevector k_x . Dashed lines correspond to predictions from the continuum model, while solid lines denote results from microstructure simulations. The winding number $v_w(\omega_b)$ around the base frequency $\omega_b = 0.5\omega_0$ transitions from -1 to $+1$, signaling a topological phase transition. (b) Full-field simulation results from the continuum model for (i) $k_c = -0.5$ and (ii) $k_c = 0.5$, with a harmonic excitation at ω_b applied at the center of the free surface. Energy localization switches between boundaries depending on the sign of k_c , illustrating the non-Hermitian skin effect. (c) Simulations performed on the microstructure, confirming the directional localization predicted by the continuum framework.

complex phase velocities and topological winding numbers. As shown in Fig. 10(c), simulations based on the exact microstructures, where the light boxes indicate the modified hexagonal lattice array, reproduce the same directional energy localization. Although the lattice is discrete, its response aligns closely with the continuum prediction, demonstrating that the skin effect persists across scales. This cross-scale consistency further validates the topological origin of the phenomenon and underscores the robustness of the proposed theoretical framework.

4.2. Polarization mode conversion

In this subsection, we investigate the transmission characteristics of elastic waves across a finite slab of odd elastic medium with thickness d , which is embedded between two semi-infinite conventional Cauchy elastic media, as illustrated in Fig. 11(a). The odd elastic layer is characterized by an effective elasticity matrix $\mathbf{K}$ and density ρ , while the surrounding Cauchy media are described by constitutive matrix $\mathbf{C}^0$ and density ρ^0 .

We consider a normally incident wave propagating perpendicular to the interfaces, with the material principal axes aligned with the global coordinate directions. Under these conditions, the incident, reflected, and transmitted waves comprise only pure longitudinal and shear components. Within the odd elastic slab, however, two distinct polarization modes can exist, whose polarization vectors, denoted as $\mathbf{P}_L$ and $\mathbf{P}_S$, are obtained by solving the eigenproblem defined in Eq. (7). These modes may exhibit elliptical or linear polarization depending on the specific form of the odd elastic tensor. Each of these polarization modes can propagate in both forward and backward directions. Accordingly, the displacement field inside the odd medium can be written as:

$$\mathbf{u}(x) = \begin{cases} \mathbf{P}_{L0}(i_L e^{(-ik_{L0}x)} + r_L e^{(ik_{L0}x)}) + \mathbf{P}_{S0}(i_S e^{(-ik_{S0}x)} + r_S e^{(ik_{S0}x)}), & x < 0 \\ \mathbf{P}_L(A e^{(-ik_Lx)} + B e^{(ik_Lx)}) + \mathbf{P}_S(C e^{(-ik_Sx)} + D e^{(ik_Sx)}), & 0 \leq x \leq d \\ \mathbf{P}_{L0}t_L e^{(-ik_{L0}x)} + \mathbf{P}_{S0}t_S e^{(-ik_{S0}x)}, & d < x \end{cases} \quad (44)$$

For the surrounding Cauchy media, the polarization vectors simplify to $\mathbf{P}_{L0} = [1, 0]^T$ for longitudinal waves and $\mathbf{P}_{S0} = [0, 1]^T$ for shear waves. In the expression for the displacement field, k_L and k_S denote the wavenumbers corresponding to the two polarization modes within the odd elastic medium. The transmission and reflection coefficients for longitudinal waves are denoted as t_L and r_L , respectively, while t_S and r_S represent those for shear waves. These four coefficients can be systematically computed using the transfer matrix method (Kim et al., 2018; Qu et al., 2022). In general, a scattering matrix $\mathbf{S}$ can be introduced to relate the amplitudes

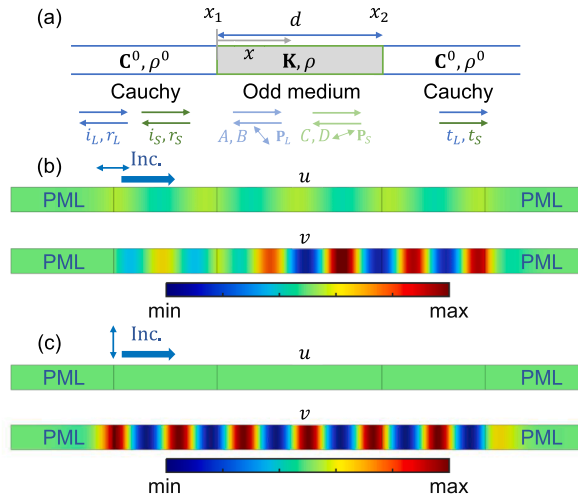


Fig. 11. Polarization-dependent wave response in the odd elastic medium. (a) Schematic illustration of polarization-dependent mode conversion behavior. (b,c) Numerical results showing that only longitudinal wave incidence leads to shear mode amplification, while shear wave incidence results in no mode conversion or amplification.

of the incident, reflected, and transmitted wave components on both sides of the layered system:

$$\begin{bmatrix} t_L \\ 0 \\ t_S \\ 0 \end{bmatrix} = \mathbf{S} \begin{bmatrix} i_L \\ r_L \\ i_S \\ r_S \end{bmatrix}. \quad (45)$$

In the above formulation, i_L and i_S represent the amplitudes of the incident longitudinal and shear waves, respectively. By specifying the incident wave as either purely longitudinal $(i_L, i_S) = (1, 0)$ or purely shear $(i_L, i_S) = (0, 1)$, the corresponding transmission and reflection coefficients can be readily computed. These coefficients characterize the wave conversion and amplification behaviors induced by the odd elastic layer under different polarization excitations.

$$\begin{cases} \mathbf{S} = \mathbf{M}_0^{-1} \cdot \mathbf{T} \cdot \mathbf{M}_0 \\ \mathbf{T} = \mathbf{M} \cdot \mathbf{N}(d) \cdot \mathbf{M}^{-1} \end{cases} \quad (46)$$

Here, $\mathbf{T}$ denotes the transfer matrix of the odd elastic layer, while $\mathbf{M}_0$ and $\mathbf{M}$ represent the impedance matrices of the background Cauchy medium and the odd elastic medium, respectively. The matrix $\mathbf{N}(d)$ is a spatial phase shift matrix that captures the propagation effects across the layer of thickness d . The explicit forms of these matrices are obtained by extracting the relevant coefficients from the governing dynamic equations.

As previously discussed, odd elastic media possess polarization-dependent amplification capabilities. To illustrate this characteristic, we consider an odd elasticity matrix of the form:

$$\mathbf{K}_{\text{eff}} = \begin{bmatrix} 2k_{ss} & 0 & 0 & 0 \\ 0 & 2k_{ss} & 0 & 0 \\ -4k_c & 0 & k_{ss} & k_{ss} \\ 0 & 0 & k_{ss} & k_{ss} \end{bmatrix}. \quad (47)$$

For the degenerate case where the polarization modes satisfy $\mathbf{P}_L = \mathbf{P}_S = [0, 1]^T$, we analyze the response of the medium to different types of incident waves based on the energy function defined in Eq. (19). In the case of a longitudinal incident wave, where the displacement polarization satisfies $\alpha_1 = 0$ and $\alpha_2 \neq 0$. The energy exhibits quadratic amplification due to coupling with the degenerate polarization direction. In contrast, for a shear incident wave with $\alpha_1 = \alpha_2 = 0$, the energy remains unchanged, since the excitation does not project onto the amplifying polarization mode.

This polarization-dependent energy behavior is further confirmed through continuum-scale simulations, as illustrated in Fig. 11(b), (c). For longitudinal wave incidence, the longitudinal component maintains constant energy, while the shear component undergoes quadratic amplification as a result of mode conversion. In the case of shear wave incidence, the shear energy remains conserved and no mode conversion is observed, consistent with the predicted polarization selectivity of the odd elastic medium. These results highlight the distinctive ability of odd elastic media to selectively amplify elastic waves based on their polarization. The degeneracy in the elasticity matrix ensures that energy amplification occurs only when the incident wave excites components aligned with the amplifying eigenmode. In the case of longitudinal wave incidence, this coupling results in significant energy gain in the orthogonal shear channel, demonstrating polarization-dependent mode conversion. Conversely, in the absence of such coupling, shear wave incidence results in no energy exchange or conversion, highlighting the anisotropic and non-reciprocal response of the medium. These findings reveal the

potential of odd elasticity in designing advanced metamaterials for directional wave manipulation, polarization-dependent filtering, and energy localization, where controlled amplification or suppression of specific modes is required.

5. Summary

This study presents a comprehensive investigation into the non-conservative mechanics of odd elastic media and their potential for advanced elastic wave manipulation. By leveraging the generalized Stokes theorem, we elucidate the physical significance of the antisymmetric components in the odd elastic tensor, thereby providing deeper theoretical insights into the governing principles of non-Hermitian elasticity. A multiscale homogenization framework is systematically developed to derive the effective continuum properties of various 2D odd lattice architectures. This approach establishes a direct and quantitative link between microscale structural design and emergent macroscopic behavior. Building on this foundation, we explore the dynamic characteristics of representative non-Hermitian systems. Through a combination of analytical derivations and numerical simulations, we demonstrate that odd elasticity enables unprecedented wave phenomena, including surface wave control, energy localization via the skin effect, and polarization-dependent mode conversion. These behaviors are validated using specially engineered odd lattices, illustrating practical pathways for engineering non-conservative mechanical responses. This work integrates non-Hermitian mechanics into elastic metamaterial design, establishing a foundation for active media that enable controllable, directional, and anisotropic wave propagation beyond the limitations of conventional elastic systems.

CRedit authorship contribution statement

Wen Cheng: Writing – review & editing, Writing – original draft, Visualization, Validation, Software, Methodology, Investigation, Formal analysis, Conceptualization; **Honghua Qian:** Writing – review & editing, Writing – original draft, Visualization, Validation, Methodology, Investigation, Formal analysis; **Shaoyun Wang:** Writing – review & editing, Writing – original draft, Validation, Investigation, Formal analysis; **Fan Yang:** Writing – original draft, Visualization, Validation, Methodology, Formal analysis, Conceptualization; **Quan Wang:** Writing – review & editing, Validation, Methodology, Conceptualization; **Rui Zhu:** Writing – review & editing, Methodology, Investigation, Conceptualization; **Gengkai Hu:** Writing – review & editing, Methodology, Conceptualization; **Guoliang Huang:** Writing – review & editing, Writing – original draft, Validation, Supervision, Methodology, Funding acquisition, Conceptualization.

Data availability

No data was used for the research described in the article.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Appendix A. Causality in odd elasticity and Kramers-Kronig relations

In this section, we discuss the principle of causality and the associated Kramers-Kronig relations. In our main analysis, the constitutive relation is taken to be constant due to the long-wavelength assumption, under which the Kramers-Kronig relations become trivial. However, realizing an odd spring requires active control, as illustrated in [Fig. A.1](#), the host beam incorporates feed-forward

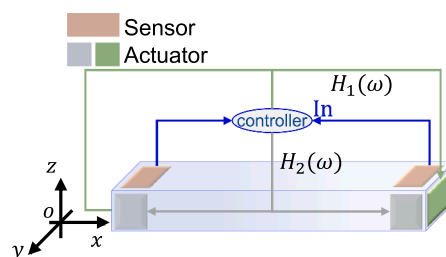


Fig. A.1. Schematic of the odd spring mechanism implemented through a feedback circuit.

control architectures based on sensor-actuator loops. In particular, piezoelectric-patch-based feed-forward control establishes a prescribed coupling between the sensed axial (x -direction) tensile deformation and the actively induced shear and tensile deformations along different directions, as governed by the transfer functions $H_1(\omega)$ and $H_2(\omega)$ (Wang et al., 2024a). Under this configuration, the odd coupling coefficient becomes dispersive, that is, $k_c = k_c(\omega)$, with its frequency dependence originating from the dynamics of the piezoelectric actuation. Consequently, the effective stiffness becomes frequency-dependent, and its dispersive behavior is fully characterized by the transfer functions $H_{1,2}(\omega)$, which encode the underlying dynamic response of the system. In such cases, it is natural to formulate the problem in the frequency domain, where the Kramers-Kronig relations arise directly from the analytic structure of the frequency-dependent modulus:

$$\sigma(\omega) = \mathbf{K}(\omega) \cdot \varepsilon(\omega), \quad (\text{A.1})$$

$$\varepsilon(\omega) = \mathbf{L}(\omega) \cdot \sigma(\omega). \quad (\text{A.2})$$

Here, $\sigma(\omega)$ and $\varepsilon(\omega)$ denote the Fourier transforms of the stress and strain fields, respectively. The matrices $\mathbf{K}(\omega)$ and $\mathbf{L}(\omega)$ are the effective complex-valued modulus and compliance, respectively, obtained from microstructure-level analysis, and they are inverses of each other. The corresponding Fourier integrals in both time and frequency domains can be written as follows:

$$\sigma(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \sigma(\omega) e^{-i\omega t} d\omega, \quad (\text{A.3})$$

and

$$\varepsilon(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \varepsilon(t') e^{i\omega t'} dt'. \quad (\text{A.4})$$

with corresponding equations for σ . The substitution of Eq. (A.1) into Eq. (A.4) with the use of Eq. (A.3) of σ gives:

$$\varepsilon(t) = \mathbf{L}_0 \cdot \sigma(t) + \int_{-\infty}^{\infty} \mathbf{J}(\tau) \cdot \sigma(t - \tau) d\tau, \quad (\text{A.5})$$

where

$$\mathbf{J}(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} [\mathbf{L}(\omega) - \mathbf{L}_0] e^{-i\omega t} d\omega. \quad (\text{A.6})$$

To ensure a causal material response, the convolution kernel in Eq. (A.6) is required to vanish for $t < 0$, and Eq. (A.5) becomes:

$$\varepsilon(t) = \mathbf{L}_0 \cdot \sigma(t) + \int_0^{\infty} \mathbf{J}(\tau) \cdot \sigma(t - \tau) d\tau, \quad (\text{A.7})$$

From Eq. (A.6), the frequency-dependent modulus (analogous to the dielectric constant) can be expressed in terms of the causal response kernel $\mathbf{J}(\tau)$ as

$$\mathbf{L}(\omega) = \mathbf{L}_0 + \int_0^{\infty} \mathbf{J}(\tau) e^{i\omega\tau} d\tau, \quad (\text{A.8})$$

where $\mathbf{J}(\tau) = 0$ for $\tau < 0$. Following the reasoning in electromagnetic theory, the analyticity of $\mathbf{L}(\omega)$ in the upper half of the complex- ω plane is guaranteed by the physical condition that $\mathbf{J}(\tau)$ is finite for all τ and vanishes as $\tau \rightarrow \infty$. For large $|\omega|$, the Laplace integral can be expanded asymptotically. Assuming a physical solid cannot respond instantaneously, i.e., $\mathbf{J}(0) = \mathbf{0}$, we obtain:

$$\mathbf{L}(\omega) - \mathbf{L}_0 = -\frac{\mathbf{J}'(0)}{\omega^2} + \mathcal{O}(\omega^{-3}), \quad (\text{A.9})$$

so that each component of $\mathbf{L}(\omega)$ decays at least as ω^{-2} along any ray into the upper half-plane. This decay guarantees that the contribution from the large-radius arc in the Kramers-Kronig contour integral vanishes. Finally, the asymptotic behavior leads to the high-frequency limit $\lim_{|\omega| \rightarrow \infty} \mathbf{L}(\omega) = \mathbf{L}_0$, corresponding to the instantaneous modulus, which is physically constrained to a finite value. Together, these three properties complete the elastic analogue of the standard proof for analyticity and asymptotic behavior of $\mathbf{L}(\omega) - \mathbf{L}_0$ (Jackson, 2012).

Because $\mathbf{L}(\omega)$ is analytic in the upper half of the complex- ω plane and approaches a constant $\mathbf{L}_0$ at high frequency, the function $\mathbf{L}(\omega) - \mathbf{L}_0$ satisfies the conditions for the Cauchy integral theorem (Jackson, 2012). Thus, we have:

$$\mathbf{L}(\omega) - \mathbf{L}_0 = \frac{1}{i\pi} \mathcal{P} \int_{-\infty}^{\infty} \frac{\mathbf{L}(\omega') - \mathbf{L}_0}{\omega' - \omega} d\omega', \quad (\text{A.10})$$

where $\mathcal{P}$ denotes the Cauchy principal value, and Eq. (A.10) therefore leads directly to the Kramers-Kronig relations that link the real and imaginary components of the dispersive modulus:

$$\begin{aligned} \text{Re}(\mathbf{L}(\omega)) &= \mathbf{L}_0 + \frac{1}{\pi} \mathcal{P} \int_{-\infty}^{\infty} \frac{\text{Im}(\mathbf{L}(\omega'))}{\omega' - \omega} d\omega', \\ \text{Im}(\mathbf{L}(\omega)) &= -\frac{1}{\pi} \mathcal{P} \int_{-\infty}^{\infty} \frac{\text{Re}(\mathbf{L}(\omega')) - \mathbf{L}_0}{\omega' - \omega} d\omega'. \end{aligned} \quad (\text{A.11})$$

The real valued nature of $\mathbf{L}(\omega)$ guarantees the symmetry condition $\mathbf{L}(-\omega) = \mathbf{L}^*(\omega)$, which implies that $\text{Re}(\mathbf{L}(\omega))$ is even and $\text{Im}(\mathbf{L}(\omega))$ is odd. Under this symmetry, the relations reduce to integrals over positive frequencies:

$$\begin{aligned}\text{Re}(\mathbf{L}(\omega)) &= \mathbf{L}_0 + \frac{2}{\pi} \mathcal{P} \int_0^\infty \frac{\omega' \text{Im}(\mathbf{L}(\omega'))}{\omega'^2 - \omega^2} d\omega', \\ \text{Im}(\mathbf{L}(\omega)) &= -\frac{2\omega}{\pi} \mathcal{P} \int_0^\infty \frac{\text{Re}(\mathbf{L}(\omega')) - \mathbf{L}_0}{\omega'^2 - \omega^2} d\omega'.\end{aligned}\quad (\text{A.12})$$

Importantly, the Kramers-Kronig relations constrain the causal structure of the constitutive matrix $\mathbf{L}(\omega)$, independent of whether the dispersion eigenmodes exhibit real or complex frequencies in the broken-phase. When the constitutive relation is constant, i.e., $\mathbf{L}(\omega) = \mathbf{L}_0$, the Kramers-Kronig relations are trivially satisfied. In this case, the modulus has no frequency dependence, and thus $\text{Im}(\mathbf{L}(\omega)) = 0$ and $\text{Re}(\mathbf{L}(\omega)) = \mathbf{L}_0$ for all ω . Substituting these into the Kramers-Kronig relations yields zero on both sides of each equation. The analyticity condition is satisfied since $\mathbf{L}(\omega)$ is constant (and therefore entire), and the high-frequency decay condition is also met as the dispersive part $\mathbf{L}(\omega) - \mathbf{L}_0$ vanishes identically. Hence, the Kramers-Kronig relations hold automatically for the nondispersive media considered in the long-wavelength limit of this study.

Overall, the Kramers-Kronig relations provide causal constraints on the frequency-dependent constitutive matrix, thereby defining an admissible class of dispersive odd-elastic models and offering practical guidance for their implementation.

Appendix B. Supplementary derivations for odd elasticity

B.1. Generalized Cauchy-Born hypothesis

We begin by revisiting the original Cauchy-Born hypothesis, which establishes a fundamental relationship between macroscopic strains and nodal displacements. When a lattice is subjected to an arbitrary deformation gradient $\mathbf{F}$, it undergoes a transformation to a deformed configuration, as expressed by:

$$\mathbf{x}(\mathbf{X}_c + \mathbf{a}) = \mathbf{x}(\mathbf{X}_c) + \mathbf{F} \cdot \mathbf{a}, \quad (\text{B.1})$$

where $\mathbf{X}_c$ and $\mathbf{a}$ denote the boundary location and lattice vector, respectively, while $\mathbf{x}$ represents the position in the reference configuration (as illustrated in Fig. B.1). The displacement field can then be written explicitly as:

$$\begin{aligned}\mathbf{u}(\mathbf{X}_c + \mathbf{a}) &= \mathbf{x}(\mathbf{X}_c + \mathbf{a}) - (\mathbf{X}_c + \mathbf{a}) \\ &= (\mathbf{x}(\mathbf{X}_c) - \mathbf{X}_c) + (\mathbf{F} - \mathbf{I}) \cdot \mathbf{a}.\end{aligned}\quad (\text{B.2})$$

We define the affine displacement as $\mathbf{u}_{\text{aff}} = \mathbf{x}(\mathbf{X}_c) - \mathbf{X}_c$. Under the assumptions of linear elasticity and infinitesimal deformations, the displacement field takes the form:

$$\mathbf{u}(\mathbf{X}_c + \mathbf{a}) = \mathbf{u}_{\text{aff}} + \nabla \mathbf{u} \cdot \mathbf{a}. \quad (\text{B.3})$$

When the stress tensor is symmetric, the antisymmetric component of the displacement gradient, which represents rigid-body rotation, does not contribute to the strain energy. In this case, the strain energy remains unchanged under a macroscopic antisymmetric deformation, and the generalized Cauchy-Born hypothesis reduces to the classical form,

$$\mathbf{u}(\mathbf{X}_c + \mathbf{a}) = \mathbf{u}_{\text{aff}} + (\nabla \mathbf{u} + \mathbf{u} \nabla)/2 \cdot \mathbf{a}. \quad (\text{B.4})$$

Conversely, when the energy contribution associated with rigid-body rotation cannot be neglected, as occurs in odd elasticity, a more general form of the Cauchy-Born relation is required.

B.2. Stress from rigid-body rotation in odd elasticity

In classical Cauchy elasticity, a rigid-body rotation does not generate stress. In odd elastic media, however, the presence of active, non-Hermitian couplings enables rigid-body rotation to produce nontrivial stress components. This appendix presents a concrete example using the square lattice of odd springs introduced in the main text to illustrate this mechanism.

We begin with the macroscopic strain vector, $\mathbf{s} = (-1, -1, 1, -1)^T$ which can be decomposed into a rigid rotation ($\mathbf{s}_1 = (0, 0, 1, -1)^T$) and a bulk component ($\mathbf{s}_2 = (-1, -1, 0, 0)^T$). The corresponding deformation fields in the continuum and lattice descriptions are shown

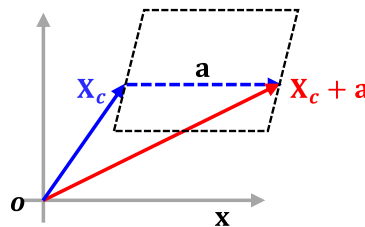


Fig. B.1. Illustration of the reference configuration.

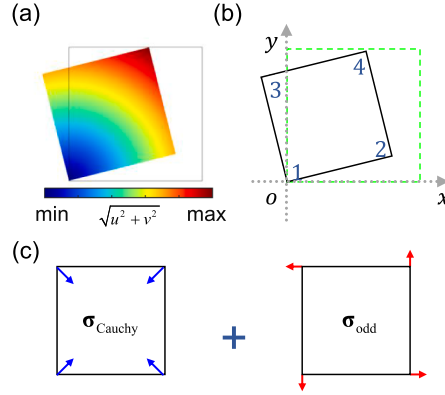


Fig. B.2. Deformation and stress in a square lattice. (a) Continuum and (b) microstructural displacement field. (c) Decomposition of the macroscopic stress into Cauchy and odd elastic contributions.

in Fig. B.2(a, b). Based on the constitutive law of the odd springs described in the main text, and specializing to the case $k_{rr} = 0$, the constraint forces at the four lattice nodes are obtained as:

$$\begin{aligned} \mathbf{f}_1 &= [k_{ss}, k_{ss} - 2k_c]^T, \\ \mathbf{f}_2 &= [-k_{ss} + 2k_c, k_{ss}]^T, \\ \mathbf{f}_3 &= [-k_{ss}, -k_{ss} + 2k_c]^T, \\ \mathbf{f}_4 &= [k_{ss} - 2k_c, -k_{ss}]^T. \end{aligned} \quad (\text{B.5})$$

From these nodal reactions, the equivalent macroscopic stress at the continuum level is evaluated as:

$$\boldsymbol{\sigma} = \frac{1}{V_{\text{area}}} \sum_{i=1}^4 \mathbf{x}_i \otimes \mathbf{f}_i = 2 \begin{bmatrix} -k_{ss} + k_c & k_c \\ -k_c & -k_{ss} + k_c \end{bmatrix}. \quad (\text{B.6})$$

Where V_{area} is the area of the lattice cell and $\mathbf{x}_i$ denotes the position vector of node i in the reference configuration. This stress naturally decomposes into a conventional Cauchy contribution (hydrostatic part) and an odd contribution (torsional part), as shown schematically in Fig. B.2(c):

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}_{\text{Cauchy}} + \boldsymbol{\sigma}_{\text{odd}} = 2 \begin{bmatrix} -k_{ss} & 0 \\ 0 & -k_{ss} \end{bmatrix} + 2 \begin{bmatrix} k_c & k_c \\ -k_c & k_c \end{bmatrix}. \quad (\text{B.7})$$

The second term represents a torsional stress arising from the stretch-twist coupling characteristic of odd springs. In a classical Cauchy elastic medium ($k_c = 0$), this antisymmetric contribution is absent from the constitutive law and therefore the rigid-body rotation produces no stress ($\boldsymbol{\sigma}_{\text{odd}} = \mathbf{0}$). In the odd lattice, however, the active coupling converts the antisymmetric component of the deformation into finite tangential forces that homogenize into a nonzero torsional stress. As a result, a rigid-body rotation that is stress-free in conventional elasticity produces nontrivial stress in the presence of odd elasticity.

B.3. Explicit formulation of the area manifold

In this subsection, we present the explicit computation of the components of the area manifold $\mathbf{A}_\varepsilon$ associated with the closed curve ℓ in the 4D strain space. As defined in the main text, $\mathbf{A}_\varepsilon$ collects the oriented projected areas of ℓ onto the 2D coordinate planes, as illustrated in Fig. 1(c). Each component $\mathbf{A}_\varepsilon(i, j)$ corresponds to the oriented area enclosed by the projection of the strain path onto the plane spanned by the strain coordinates $\varepsilon(i)$ and $\varepsilon(j)$. These quantities are obtained by integrating the corresponding strain pairs along the closed path ℓ . The explicit expressions are:

$$\begin{aligned} \mathbf{A}_\varepsilon(1, 2) &= \frac{1}{2} \oint_\ell \varepsilon_{11} d\varepsilon_{22} - \varepsilon_{22} d\varepsilon_{11}, \\ \mathbf{A}_\varepsilon(1, 3) &= \frac{1}{2} \oint_\ell \varepsilon_{11} d\varepsilon_{12} - \varepsilon_{12} d\varepsilon_{11}, \\ \mathbf{A}_\varepsilon(1, 4) &= \frac{1}{2} \oint_\ell \varepsilon_{11} d\varepsilon_{21} - \varepsilon_{21} d\varepsilon_{11}, \\ \mathbf{A}_\varepsilon(2, 3) &= \frac{1}{2} \oint_\ell \varepsilon_{22} d\varepsilon_{12} - \varepsilon_{12} d\varepsilon_{22}, \\ \mathbf{A}_\varepsilon(2, 4) &= \frac{1}{2} \oint_\ell \varepsilon_{22} d\varepsilon_{21} - \varepsilon_{21} d\varepsilon_{22}, \\ \mathbf{A}_\varepsilon(3, 4) &= \frac{1}{2} \oint_\ell \varepsilon_{12} d\varepsilon_{21} - \varepsilon_{21} d\varepsilon_{12}. \end{aligned} \quad (\text{B.8})$$

These quantities characterize the oriented areas enclosed by the strain-path projections and form the geometric basis for evaluating path-dependent work in odd elasticity.

Appendix C. Effective models for triangular and modified hexagonal lattices

In the main text, we established a continuum model for square lattice composed of odd springs, highlighting their dynamic and non-conservative mechanical behavior. To demonstrate the generality and versatility of the proposed homogenization framework, we now extend the analysis to two additional canonical lattice geometries: triangular and modified hexagonal lattices. These configurations are commonly encountered in crystalline solids and architected metamaterials, where accurate prediction of effective properties is critical for applications such as elastic wave control, topological mechanics, and programmable material design.

The corresponding lattice structures are illustrated in Fig. C.1(a), each characterized by a lattice constant L_0 and a set of lattice vectors $\mathbf{a}_1$ and $\mathbf{a}_2$. Independent and dependent nodes are marked in blue and orange, respectively. By employing the same multiscale homogenization strategy presented in the main text, which is based on the Cauchy-Born hypothesis and energy averaging at the unit-cell level, we derive the effective constitutive tensors for each configuration.

For the triangular lattice, the resulting effective elasticity matrix takes the form:

$$\mathbf{K}_{\text{Tri}} = \frac{\sqrt{3}}{4} \begin{bmatrix} 3k_{ss} + k_{rr} & k_{ss} - k_{rr} & 4k_c & 0 \\ k_{ss} - k_{rr} & 3k_{ss} + k_{rr} & 0 & -4k_c \\ -4k_c & 0 & k_{ss} + 3k_{rr} & k_{ss} - k_{rr} \\ 0 & 4k_c & k_{ss} - k_{rr} & k_{ss} + 3k_{rr} \end{bmatrix}. \quad (\text{C.1})$$

Similarly, for the modified hexagonal lattice, the effective elasticity matrix is expressed as:

$$\mathbf{K}_{\text{Hex}} = \frac{\sqrt{3}}{6} \begin{bmatrix} 11k_{ss} + 3k_{rr} & 5k_{ss} - 3k_{rr} & 16k_c & 0 \\ 5k_{ss} - 3k_{rr} & 11k_{ss} + 3k_{rr} & 0 & -16k_c \\ -16k_c & 0 & 3k_{ss} + 11k_{rr} & 3k_{ss} - 5k_{rr} \\ 0 & 16k_c & 3k_{ss} - 5k_{rr} & 3k_{ss} + 11k_{rr} \end{bmatrix}. \quad (\text{C.2})$$

Building on these continuum models, we investigate the dynamic behavior of the triangular and modified hexagonal lattices by comparing their dispersion relations with those computed from fully resolved discrete calculations. In the following examples, we set the odd parameters to $k_{ss} = 1, k_{rr} = 0.8$ and $k_c = 0.5$ for illustration. The results are shown in Fig. C.1(b), with panels (i) and (ii) corresponding to the triangular and modified hexagonal cases, respectively. The continuum predictions (solid lines) align closely with the discrete results (dots) in the long-wavelength limit, capturing both the real and imaginary components of the complex

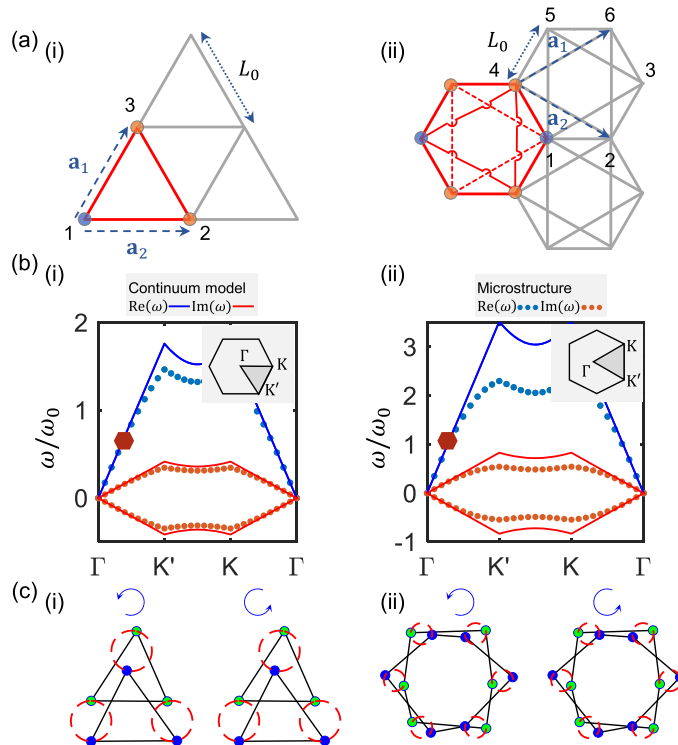


Fig. C.1. Dynamic behavior of triangular and modified hexagonal lattices. (a) Unit cell geometries of (i) triangular and (ii) modified hexagonal lattices. (b) Dispersion relations from the continuum model (solid lines) and discrete simulations (dots), showing excellent agreement in the long-wavelength limit. Shaded areas denote the IBZ. (c) Representative rotational modes at low frequencies corresponding to (b), illustrating the underlying microscale mechanisms.

band structure. Dispersion is evaluated along the boundaries of the irreducible Brillouin zone (IBZ), which are shaded in gray. To further elucidate the microscale mechanisms underlying the observed dynamics, Fig. C.1(c) displays representative rotational modes associated with low-frequency points in the dispersion diagrams. These mode shapes highlight the localized rotational behavior that gives rise to the system's nontrivial dispersion features.

The close correspondence between continuum and discrete models across multiple lattice symmetries confirms the accuracy and generality of the proposed homogenization approach. Beyond reproducing low-frequency dispersion characteristics, the model faithfully captures key modal features, offering a robust and broadly applicable framework for analyzing wave propagation in odd elastic metamaterials.

Appendix D. Directional energy transport in elastic wave propagation

In the main text, we demonstrated directional energy transport along both the primary and secondary diagonals using an odd elastic medium. To further highlight the generality and versatility of the proposed mechanism, this appendix introduces additional design strategies for achieving directional transport in other orientations.

Consider the lattice structure illustrated in Fig. D.1(a), where the gray lines represent conventional springs with longitudinal modulus k_{ss} , and the blue lines denote odd springs incorporating both longitudinal modulus k_{ss} and coupling modulus k_c . Using the homogenization framework introduced earlier, the effective elasticity matrix $\mathbf{K}_T$ corresponding to this rotated coupling configuration can be derived as:

$$\mathbf{K}_T = \begin{bmatrix} 3k_{ss} & k_{ss} - k_c & k_c & 0 \\ k_{ss} + k_c & 3k_{ss} & 0 & -k_c \\ -k_c & 0 & k_{ss} & k_{ss} + k_c \\ 0 & k_c & k_{ss} - k_c & k_{ss} \end{bmatrix}. \quad (\text{D.1})$$

At the normalized frequency $\omega/\omega_0 = 0.25$, the isofrequency contours corresponding to different coupling strengths are shown in Fig. D.1(b), (c). Subfigure (b) presents the real part of the dispersion relations, while subfigure (c) displays their imaginary counterparts. Panels (i), (ii), and (iii) correspond to increasing values of the coupling coefficient k_c . Solid lines represent the continuum model predictions, and circular markers denote the numerical results from the discrete microstructure. Excellent agreement is observed between the two approaches in the long-wavelength regime, confirming the accuracy of the homogenization framework. Following the classification scheme established in the main text, several key features emerge from the simulated field patterns, closely paralleling those observed in the isofrequency contours:

- **Case (i):** In the unbroken-phase regime, the system exhibits no pronounced spatial localization. Wave propagation remains anisotropic but relatively uniform, with no dominant gain or attenuation directions.
- **Case (ii):** Two EPs appear along the $-\pi/4$ and $3\pi/4$ directions, where the real parts of the dispersion branches coalesce. As previously discussed, these directions are associated with quadratic amplification, which is a characteristic feature of EP-induced wave dynamics.
- **Case (iii):** In the broken-phase regime, complex-valued dispersion relations emerge near the $-\pi/4$ and $3\pi/4$ directions, signifying exponential amplification and attenuation along these specific paths. This leads to strong directional selectivity in energy transport.

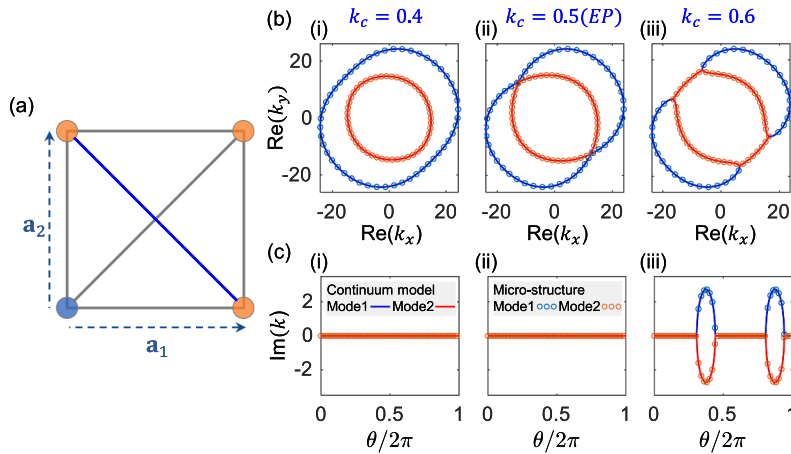


Fig. D.1. Directional wave behavior in odd elastic lattices with rotated coupling configuration, highlighting gain along the secondary diagonals. (a) Microstructure composed of both conventional springs and odd springs. (b) Real and (c) imaginary parts of the isofrequency contours at normalized frequency $\omega = 0.25$, shown for three representative coupling strengths: (i) $k_c = 0.4$, (ii) 0.5 , and (iii) 0.6 .

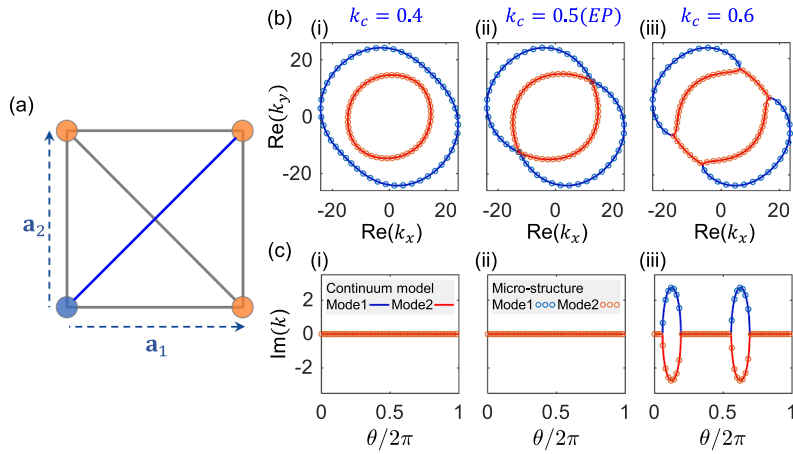


Fig. D.2. Directional wave behavior in odd elastic lattices with original coupling configuration, highlighting gain along the principal diagonals. (a) Microstructure consisting of both conventional springs and odd springs. (b) Real and (c) imaginary parts of the isofrequency contours at the normalized frequency $\omega = 0.25$, shown for three representative values of the coupling coefficient: (i) $k_c = 0.4$, (ii) 0.5 , and (iii) 0.6 .

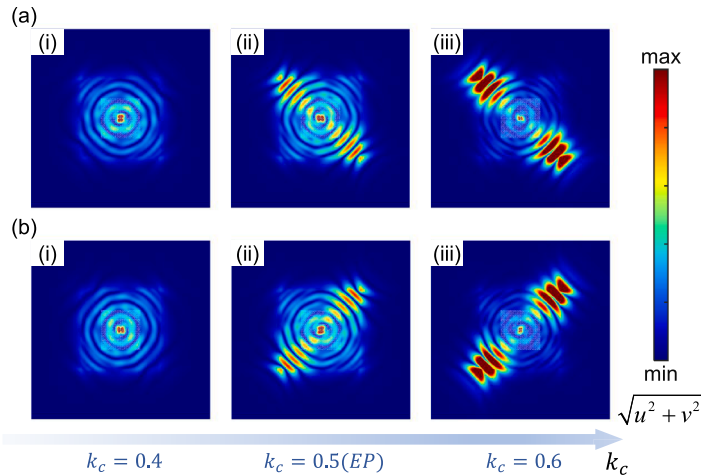


Fig. D.3. Directional control of elastic waves in odd elastic media. (a) Displacement fields exhibiting off-diagonal gain, and (b) displacement fields exhibiting diagonal gain. Cases (i), (ii), and (iii) correspond to increasing coupling strengths, each showing distinct wave propagation behaviors.

These findings demonstrate that directional amplification of elastic waves can be systematically achieved via the design of parity-breaking elastic microstructures. Moreover, tuning the coupling coefficient k_c allows precise control over the nature of wave gain, whether quadratic or exponential, and enables advanced manipulation of energy flow within the continuum framework.

To explore directional tunability in alternative orientations, we consider a modified elasticity matrix:

$$\mathbf{K}_B = \begin{bmatrix} 3k_l & k_l + k_c & k_c & 0 \\ k_l - k_c & 3k_l & 0 & -k_c \\ -k_c & 0 & k_l & k_l - k_c \\ 0 & k_c & k_l + k_c & k_l \end{bmatrix}. \quad (\text{D.2})$$

This configuration can be physically realized by reconfiguring the placement of the odd springs, as illustrated in Fig. D.2(a). At the same normalized frequency $\omega = 0.25$, the corresponding isofrequency contours derived from both the continuum model and discrete simulations are shown in Fig. D.2(b), (c). As in previous cases, strong agreement is observed in the long-wavelength regime, validating the effectiveness of the homogenization framework.

Importantly, this modified configuration results in a rotation of the directions exhibiting wave localization and amplification, now oriented along approximately $\pi/4$ and $-3\pi/4$. This directional shift highlights the structural tunability of the odd elastic lattice and further emphasizes its versatility in engineering customized wave propagation behavior.

To further illustrate the directional energy transport facilitated by the designed odd elastic medium, full-wave simulations were conducted using a monopole source placed at the center of a 40×40 odd elastic lattice. The resulting displacement field contours for three representative cases are presented in Fig. D.3. In case (i), corresponding to unbroken-phase regime, Fig. D.3(a), (b)–(i) shows

weak spatial localization and broadly anisotropic wave propagation, with no significant directional amplification. This behavior is consistent with the real valued isofrequency contours previously discussed. In case (ii), depicted in Fig. D.3(a),(b)–(ii), the system approaches EPs along the $-\pi/4$ and $3\pi/4$ directions. This leads to pronounced beam formation along these diagonals and quadratic energy amplification during outward wave propagation. In case (iii), illustrated in Fig. D.3(a), (b)–(iii), the system enters the broken-phase regime. The displacement field becomes sharply localized along the $-\pi/4$ and $3\pi/4$ directions, where exponential amplification dominates. Other directions exhibit strong attenuation, resulting in highly focused, directional energy beams.

Overall, these results collectively confirm that directional wave transport and amplification in elastic media can be precisely tailored via structural design. By tuning the distribution and strength of odd couplings, the system transitions smoothly between regimes of isotropic propagation, directionally selective quadratic gain, and sharply localized exponential amplification. This establishes a robust and tunable platform for programmable wave control in non-Hermitian elastic systems, thereby reinforcing the central conclusions of the main text.

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