

# THE FREQUENCY EQUATION OF THICKNESS-SHEAR VIBRATIONS OF SC-CUT QUARTZ CRYSTAL PLATES

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The thickness-shear vibrations of a quartz crystal plate serve as the functioning mode of a resonator with strong couplings to many other modes of vibrations, which can affect the frequency and mode shape, consequently the resonator properties. For applications, it is always desired to have pure thickness-shear vibrations that only exist in an infinite plate. With the three-dimensional equations of elasticity in Cartesian coordinates, boundary conditions are approximated and the frequency equation of the thickness-shear mode is given in terms of materials constants and plate parameters. This relation shows that the thickness-shear frequency can be determined with known parameters of plates. The frequency equation then is compared with the frequency spectra from the Mindlin plate equations for the determination of some key coefficients. This is an extension of our earlier study on the frequency equation of AT-cut quartz crystal plates and it will be used in the design of SC-cut quartz crystal resonators of rectangular shape.

**Keywords:** Frequency equation; Thickness-shear mode; Plate vibration; Quartz crystal resonator

## 1. INTRODUCTION

The design of quartz crystal resonators requires accurate analysis of vibrations of the quartz crystal plate with influences of some complications like electrodes and mountings. With the increasingly miniaturization of resonators and relative increase of the effects of these complications, the challenge of analysis of vibration of quartz crystal resonator structure is increasingly tough. Particularly, with rise of vibration frequency and consequently increase of computational cost of time, empirical procedure based on proper selection of trial parameters are also in great demand. From our discussions with design engineers, a better prediction of vibration frequency and even the resonator properties with known parameters is urgently needed at least for the prototyping. To meet such needs, we started the resonator modeling with structural parameters by approximating the fundamental thickness-shear (TSh) frequency with a simple equation for the AT-cut quartz crystal plates [1]. It turned out such an approximation is quite accurate and useful in the initial selection of structural parameters. The essence of the procedure is to approximate the boundary conditions and the equations of motion of the dominant mode and a simple relation with the thickness and aspect ratios of the AT-cut quartz crystal plate is given in agreement with earlier results. As a further validation, the relation was confirmed from the straight-crested wave solutions with excellent agreement. Design engineers also agreed that such

results are useful in their resonator development process. With this successful outcome, we are now ready for the extension of the frequency equation to the SC-cut quartz crystal plates that have stronger couplings of vibration modes.

The vibrations of finite elastic plates are actually difficult to solve with various boundary conditions and wide frequency range. Fortunately, we only have interest in some special vibration modes e.g. thickness-shear mode in resonator applications, so the problem can be simplified and solved. For instance, the resonant frequency of thickness-shear modes in an infinite plate were found by Koga [2] in closed form, but corresponding simple solutions are not obtained for finite plate. By using the perturbation method, Ekstein [3] approximately obtained the frequency equation of rectangular plate. To analyze the high frequency vibrations of quartz crystal plates, Mindlin developed the first-order equations to include the thickness-shear vibration mode and couplings with other modes like the flexural vibrations [4]. With the Mindlin plate equations, now it is possible to study the high frequency vibrations in a systematic manner [5].

With the introduction of the Mindlin plate theory, the straight-crested wave solutions are obtained and they have been helping design engineers with the selection of plate parameters. For plates with relatively larger length to width ratios, the straight-crested waves, implying neglecting the width effect of solutions, are adequate for typical resonator designs. Besides, the simple and

elegant equations also serve as a foundation for the development of resonator design procedure. By combining the three-dimensional approximation and Mindlin plate equations, we obtained the quadratic frequency equation which will be useful for practical applications in the resonator design.

## 2. THREE-DIMENSIONAL EQUATIONS OF VIBRATION OF PLATES

Vibrations of elastic solid are governed by three-dimension equations of elasticity, which are applicable to plates for further simplification. For a rectangular plate with coordinates shown in Fig.1, and the equations and notations are from Mindlin [5].

First, the equations of motion are

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$$\begin{aligned} \frac{\partial T_1}{\partial x_1} + \frac{\partial T_6}{\partial x_2} + \frac{\partial T_5}{\partial x_3} &= \rho \frac{\partial^2 u_1}{\partial t^2} \\ \frac{\partial T_6}{\partial x_1} + \frac{\partial T_2}{\partial x_2} + \frac{\partial T_4}{\partial x_3} &= \rho \frac{\partial^2 u_2}{\partial t^2} \\ \frac{\partial T_5}{\partial x_1} + \frac{\partial T_4}{\partial x_2} + \frac{\partial T_3}{\partial x_3} &= \rho \frac{\partial^2 u_3}{\partial t^2} \end{aligned} \quad (1)$$

where  $T_k$  ( $k = 1,2,3,4,5,6$ ),  $x_i$  ( $i = 1,2,3$ ),  $u_i$  ( $i = 1,2,3$ ) and  $t$  are stress, coordinate, displacement, and time, respectively.

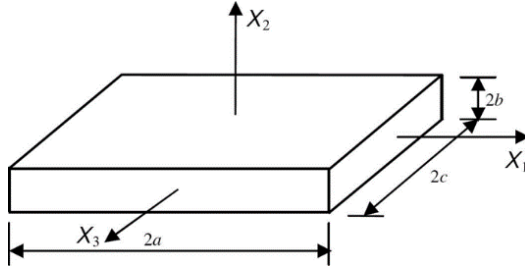


Figure 1. A rectangular plate with coordinate system

For a SC-cut quartz plate, the constitutive equations are

$$\begin{aligned} T_1 &= c_{11}S_1 + c_{12}S_2 + c_{13}S_3 + c_{14}S_4 + c_{15}S_5 + c_{16}S_6 \\ T_2 &= c_{21}S_1 + c_{22}S_2 + c_{23}S_3 + c_{24}S_4 + c_{25}S_5 + c_{26}S_6 \\ T_3 &= c_{31}S_1 + c_{32}S_2 + c_{33}S_3 + c_{34}S_4 + c_{35}S_5 + c_{36}S_6 \\ T_4 &= c_{41}S_1 + c_{42}S_2 + c_{43}S_3 + c_{44}S_4 + c_{45}S_5 + c_{46}S_6 \\ T_5 &= c_{51}S_1 + c_{52}S_2 + c_{53}S_3 + c_{54}S_4 + c_{55}S_5 + c_{56}S_6 \\ T_6 &= c_{61}S_1 + c_{62}S_2 + c_{63}S_3 + c_{64}S_4 + c_{65}S_5 + c_{66}S_6 \end{aligned} \quad (2)$$

where  $c_{st}$  ( $s, t = 1,2,3,4,5,6$ ) are the elastic constants and strain  $S_k$  ( $k = 1,2,3,4,5,6$ ) are

$$\begin{aligned} S_1 &= \frac{\partial u_1}{\partial x_1}, & S_2 &= \frac{\partial u_2}{\partial x_2}, & S_3 &= \frac{\partial u_3}{\partial x_3} \\ S_4 &= \frac{\partial u_3}{\partial x_2} + \frac{\partial u_2}{\partial x_3}, & S_5 &= \frac{\partial u_3}{\partial x_1} + \frac{\partial u_1}{\partial x_3}, & S_6 &= \frac{\partial u_2}{\partial x_1} + \frac{\partial u_1}{\partial x_2} \end{aligned}$$

(3)

By substituting (3) and (2) into (1), we have the equation of motion represented in displacements

$$\begin{aligned} &c_{11}u_{1,11} + c_{12}u_{2,21} + c_{13}u_{3,31} + c_{14}(u_{3,21} + u_{2,31}) \\ &\quad + c_{15}(u_{3,11} + u_{1,31}) + c_{16}(u_{2,11} + u_{1,21}) \\ &+ c_{61}u_{1,12} + c_{62}u_{2,22} + c_{63}u_{3,32} + c_{64}(u_{3,22} + u_{2,32}) \\ &\quad + c_{65}(u_{3,12} + u_{1,32}) + c_{66}(u_{2,12} + u_{1,22}) \\ &+ c_{51}u_{1,13} + c_{52}u_{2,23} + c_{53}u_{3,33} + c_{54}(u_{3,23} + u_{2,33}) \\ &\quad + c_{55}(u_{3,13} + u_{1,33}) + c_{56}(u_{2,13} + u_{1,23}) = \rho \frac{\partial^2 u_1}{\partial t^2} \end{aligned} \quad (4)$$

$$\begin{aligned} &c_{61}u_{1,11} + c_{62}u_{2,21} + c_{63}u_{3,31} + c_{64}(u_{3,21} + u_{2,31}) \\ &\quad + c_{65}(u_{3,11} + u_{1,31}) + c_{66}(u_{2,11} + u_{1,21}) \\ &+ c_{21}u_{1,12} + c_{22}u_{2,22} + c_{23}u_{3,32} + c_{24}(u_{3,22} + u_{2,32}) \\ &\quad + c_{25}(u_{3,12} + u_{1,32}) + c_{26}(u_{2,12} + u_{1,22}) \\ &+ c_{41}u_{1,13} + c_{42}u_{2,23} + c_{43}u_{3,33} + c_{44}(u_{3,23} + u_{2,33}) \\ &\quad + c_{45}(u_{3,13} + u_{1,33}) + c_{46}(u_{2,13} + u_{1,23}) = \rho \frac{\partial^2 u_2}{\partial t^2} \end{aligned} \quad (5)$$

$$\begin{aligned} &c_{51}u_{1,11} + c_{52}u_{2,21} + c_{53}u_{3,31} + c_{54}(u_{3,21} + u_{2,31}) \\ &\quad + c_{55}(u_{3,11} + u_{1,31}) + c_{56}(u_{2,11} + u_{1,21}) \\ &+ c_{41}u_{1,12} + c_{42}u_{2,22} + c_{43}u_{3,32} + c_{44}(u_{3,22} + u_{2,32}) \\ &\quad + c_{45}(u_{3,12} + u_{1,32}) + c_{46}(u_{2,12} + u_{1,22}) \\ &+ c_{31}u_{1,13} + c_{32}u_{2,23} + c_{33}u_{3,33} + c_{34}(u_{3,23} + u_{2,33}) \\ &\quad + c_{35}(u_{3,13} + u_{1,33}) + c_{36}(u_{2,13} + u_{1,23}) = \rho \frac{\partial^2 u_3}{\partial t^2} \end{aligned} \quad (6)$$

Then we assume the displacements as

$$u_j = A_j e^{i(\xi x_1 + \eta x_2 + \zeta x_3 - \omega t)} \quad (7)$$

where  $A_j$  ( $j = 1,2,3$ ) are amplitudes and  $\xi, \eta, \zeta, \omega$  are wavenumbers and frequency, respectively.

For further simplification, we define the normalized variables as

$$\begin{aligned} \Omega &= \frac{\omega}{\frac{\pi}{2b} \sqrt{\frac{c_{66}}{\rho}}}, & X &= \frac{\xi}{\frac{\pi}{2b}}, & Y &= \frac{\eta}{\frac{\pi}{2b}}, \\ Z &= \frac{\zeta}{\frac{\pi}{2b}}, & c_{ij} &= \frac{c_{ij}}{c_{66}} \end{aligned} \quad (8)$$

With (4-8), we have the Christoffel equation

$$\begin{aligned} \Omega^2 A_1 &= (c_{11}X^2 + Y^2 + c_{55}Z^2 + 2c_{16}XY + 2c_{15}XZ \\ &\quad + 2c_{56}YZ)A_1 + (c_{14} + c_{56})XZ + (c_{25} + c_{46})YZ]A_2 \\ &\quad + [c_{15}X^2 + c_{46}Y^2 + c_{35}Z^2 + (c_{14} + c_{56})XY \\ &\quad + (c_{13} + c_{55})XZ + (c_{36} + c_{46})YZ]A_3 \end{aligned} \quad (9)$$

$$\begin{aligned} \Omega^2 A_2 = & [c_{16}X^2 + c_{26}Y^2 + c_{45}Z^2 + (c_{12} + 1)XY + \\ & (c_{14} + c_{56})X + (c_{25} + c_{46})YZ]A_1 + (X^2 + c_{22}Y^2 + \\ & c_{44}Z^2 + 2c_{26}XY + 2c_{46}XZ + 2c_{24}YZ)A_2 + [c_{56}X^2 \\ & + c_{24}Y^2 + c_{34}Z^2 + (c_{45} + c_{46})XY + (c_{36} + c_{45})XZ \\ & + (c_{33} + c_{44})YZ]A_3 \end{aligned} \quad (10)$$

$$\begin{aligned} \Omega^2 A_3 = & [c_{51}X^2 + c_{46}Y^2 + c_{35}Z^2 + (c_{14} + c_{56})XY + \\ & (c_{13} + c_{55})XZ + (c_{45} + c_{36})YZ]A_1 + [c_{56}X^2 + c_{24}Y^2 \\ & + c_{34}Z^2 + (c_{25} + c_{46})XY + (c_{45} + c_{36})XZ \\ & + (c_{23} + c_{44})YZ]A_2 + [c_{55}X^2 + c_{44}Y^2 + c_{33}Z^2 \\ & + 2c_{45}XY + 2c_{35}XZ + 2c_{34}YZ]A_3 \end{aligned} \quad (11)$$

They show the relationship of frequency, wave numbers and amplitudes of travelling waves.

### 3. FREQUENCY EQUATION OF THICKNESS-SHEAR VIBRATIONS OF PLATES

Now we shall consider the boundary conditions, the relationship of wavenumber and size of plate, then substituting them into Christoffel equation to obtain frequency equation.

First, the traction-free boundary condition can be written as

$$T_1 = T_5 = T_6 = 0, \quad x_1 = \pm a \quad (12)$$

$$T_2 = T_4 = T_6 = 0, \quad x_2 = \pm b \quad (13)$$

$$T_3 = T_4 = T_5 = 0, \quad x_3 = \pm c \quad (14)$$

where  $a$ ,  $b$ ,  $c$  are the half-length, thickness and width of the plate in Fig. 1. The problem is now to find displacements, which will satisfy the traction-free boundary condition.

Apparently, there are not displacement precisely satisfy the boundary conditions and equations. Because thickness-shear mode is the major mode we are concerned in a quartz crystal resonator, we only consider the displacement of this particular mode. To satisfy the boundary conditions in a precise manner, we ignore all other displacements. As a result, we assume the displacement of thickness-shear mode [6-8] as

$$u_1 = A_1 \cos \xi x_1 \sin \eta x_2 \cos \zeta x_3 \quad (15)$$

By substituting (15) into the boundary conditions of (13), we only have the stress component  $T_6$  left for consideration. The major condition is the vanish of  $T_6$ , and other conditions can be neglected. As a result, we must have

$$-c_{61}\xi \sin \xi x_1 \sin \eta b \cos \zeta x_3 - c_{65}\zeta \cos \xi x_1 \sin \eta b * \sin \zeta x_3 + c_{66}\eta \xi \cos \xi x_1 \cos \eta b \cos \zeta x_3 = 0 \quad (16)$$

Obviously, the solution does not exist from above equation. But knowing that  $c_{61}$  and  $c_{65}$  are much

smaller than  $c_{66}$ , we can have the boundary equation further simplified to:

$$\cos \eta b = 0 \quad (17)$$

or

$$\eta = \frac{n\pi}{2b} \quad (n = 1, 3, 5 \dots) \quad (18)$$

for the antisymmetric thickness deformation. Through the normalization of (18), we have

$$Y = \frac{\eta}{\pi} = n \quad (19)$$

Now we turn to the two faces at ends of plate in the length direction. Because only displacement  $u_1$  is considered, the dominant boundary condition will now be  $T_1 = 0$ . With the constitutive relations in (2), and neglect  $c_{15}$  and  $c_{16}$  since they are much smaller than  $c_{11}$ , we have

$$\sin \xi a = 0 \quad (20)$$

or

$$\xi = \frac{m\pi}{a} \quad (m = 1, 2, 3 \dots) \quad (21)$$

Again, through normalization of (21) we have

$$X = \frac{\xi}{\pi} = \frac{2bm}{a} \quad (m = 1, 2, 3 \dots) \quad (22)$$

Third, on the two faces at ends of width direction, also only the condition  $T_5 = 0$  is required, and neglect  $c_{51}$ ,  $c_{56}$  again, we have

$$\sin \zeta c = 0 \quad (23)$$

or

$$\zeta = \frac{l\pi}{c} \quad (l = 1, 2, 3 \dots) \quad (24)$$

then the normalized wavenumber is

$$Z = \frac{\zeta}{\pi} = \frac{2bl}{c} \quad (l = 1, 2, 3 \dots) \quad (25)$$

Now we substitute (17), (20), and (23) into frequency equation (9), we have

$$\begin{aligned} \Omega^2 = & C_0 n^2 + C_1 \frac{2mnb}{a} + C_2 \left(\frac{2mb}{a}\right)^2 + C_3 \left(\frac{2mb}{a}\right) \left(\frac{2lb}{c}\right) \\ & + C_4 \frac{2nlb}{c} + C_5 \left(\frac{2bl}{c}\right)^2 \end{aligned} \quad (26)$$

where

$$C_0 = 1 + c_{26} \frac{A_2}{A_1} + c_{46} \frac{A_3}{A_1} \quad (27)$$

$$C_1 = 2c_{16} + (1 + c_{12}) \frac{A_2}{A_1} + (c_{14} + c_{56}) \frac{A_3}{A_1} \quad (28)$$

$$C_2 = c_{11} + c_{16} \frac{A_2}{A_1} + c_{15} \frac{A_3}{A_1} \quad (29)$$

$$C_3 = 2c_{15} + (c_{14} + c_{56})\frac{A_2}{A_1} + (c_{13} + c_{55})\frac{A_3}{A_1} \quad (30)$$

$$C_4 = 2c_{56} + (c_{25} + c_{46})\frac{A_2}{A_1} + (c_{36} + c_{46})\frac{A_3}{A_1} \quad (31)$$

$$C_5 = c_{55} + c_{45}\frac{A_2}{A_1} + c_{35}\frac{A_3}{A_1} \quad (32)$$

Because thickness-shear mode is the major mode in quartz resonator, its amplitude  $A_1$  should be much greater than  $A_2$ ,  $A_3$ , and  $a \gg b$ ,  $b \gg c$  in the real resonator. The right side terms except the first one in (24) can be thought as infinitesimal, so based on the Taylor expansion, the extraction of the root of (24) will be

$$\Omega = \sqrt{C_0} + \frac{1}{2\sqrt{C_0}} * \left( C_1 \frac{2mnb}{a} + C_2 \left( \frac{2mb}{a} \right)^2 + C_3 \left( \frac{2mb}{a} \right) \left( \frac{2lb}{c} \right) + C_4 \frac{2nlb}{c} + C_5 \left( \frac{2bl}{c} \right)^2 \right) \quad (33)$$

This is the vibration frequency equation of thickness-shear mode in an SC-cut quartz crystal plate, it is important in designing the quartz resonators. Furthermore, set  $m = n = 1$ ,  $l = 0$ , (33) can be simplified as

$$\Omega = B_0 + B_1 \frac{b}{a} + B_2 \left( \frac{b}{a} \right)^2 \quad (34)$$

where  $B_i$  is the coefficient in (33), which contain the ratio of amplitudes and elastic constants. From (34), the vibration frequency equation of thickness-shear mode of SC-cut quartz plates has a simple form which is a quadratic function, and it is coincident with the equation of AT-cut quartz. Through Mindlin plate theory, the frequency spectrum [9] that considered thickness-shear and flexural deformation has been calculated already in Fig. 2 with the thickness-shear mode is the dominant mode appeared in the midpoint of the flat portion, which is denoted by circles. Then, the validity of the frequency equation is confirmed by fitting those circles with the equation (34). We find the correlation coefficient R-square is equal to 0.9993, which is very close to 1, representing an excellent fitting.

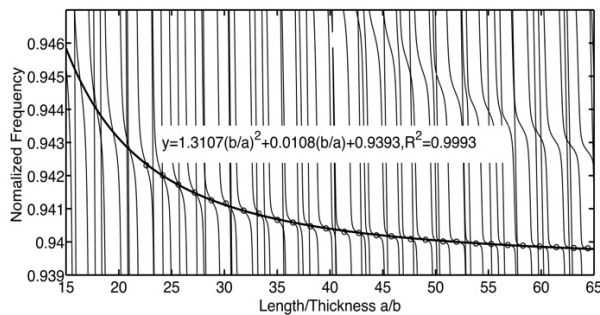


Figure 2. Frequency spectrum of a SC-cut quartz plate with thickness-shear and flexural deformation

Although the frequency equation has a simple form

to use by design engineers, some problems need to be addressed. First, this equation is suitable for plates which only consider the vibration of thickness-shear mode. Second, the size of the plate must be satisfied that length is much greater than width and the width also much greater than thickness. Third, only quartz crystal of SC-cut is considered, and the result may be changed because the different constitutive equation caused by different material constants. Finally, the real resonant frequency is discrete in Fig. 2, but it is continuous in frequency equation, so engineers need to valid the point in frequency equation whether it is real frequency by a lot of tests.

#### 4. CONCLUSIONS

We have obtained the quadratic polynomial in length to thickness ratios for the accurate estimation of the thickness-shear frequency. The frequency has been validated with accurate TSh frequencies from the coupled equations of TSh and flexural vibrations with the Mindlin plate equations. Such results will be useful in the selection of aspect ratios of SC-cut rectangular resonators.

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